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ปีการศึกษา 2558

GENERALIZED SIMPLE WAVES OF THE GAS DYNAMICS EQUATIONS ADJOINING THROUGH A SHOCK WAVE

Wiparat Worapitpong

A Thesis Submitted in Partial Fulfillment of the Requirements for the

Degree of Master of Science in Applied Mathematics

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GENERALIZED SIMPLE WAVES OF THE GAS DYNAMICS EQUATIONS ADJOINING THROUGH A SHOCK WAVE

Suranaree University of Technology has approved this thesis submitted in partial fulfillment of the requirements for a Master's Degree.

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เป็นที่ทราบแล้วว่าปัญหาของการปะทะกันของสองคลื่นรีมันน์นั้น ไม่สามารถหาผลเฉลยได้หากการหาผลเฉลยพิจารณาจากกรอบแนวคิดเพียงแค่ผลเฉลยคงตัวและคลื่นรีมันน์ ที่เป็นเช่นนั้นเนื่องด้วยเอนโทรปีของคลื่นรีมันน์มีค่าคงตัว แต่เมื่อเกิดการปะทะกันของคลื่นกระแทกและคลื่นส่วนขยาย เอนโทรปีจะไม่คงตัว เมื่อไม่นานมานี้ได้มีการพบคลาสใหม่ของผลเฉลยของสมการพลศาสตร์แก๊ส และเรียกคลาสของผลเฉลยนี้ว่า คลาสของคลื่นเชิงเดี่ยวที่ถูกวางนัยทั่วไป ลักษณะหลักของผลเฉลยเหล่านี้ คือ ไม่เป็นเอนโทรปีเท่า ขณะที่สมบัติอื่น ๆ ยังคงมีความคล้ายคลึงกับคลื่นเชิงเดี่ยว ในวิทยานิพนธ์นี้ได้ศึกษาสองคลื่นเชิงเดี่ยวที่ถูกวางนัยทั่วไป ซึ่งถูกแยกออกโดยคลื่นกระแทก

ในการหาผลเฉลยจะถูกแบ่งออกเป็น 2 ขั้นตอนดังนี้ สำหรับการหาผลเฉลยของปัญหาอันตรกิริยาของสองคลื่นเชิงเดี่ยวที่ถูกวางนัยทั่วไปผ่านคลื่นกระแทกนั้น ได้เขียนโปรแกรมภายใต้โปรแกรม MATLAB เพื่อแก้ปัญหา จากนั้นใช้โปรแกรม REDUCE หาฟังก์ชันด้านขวาของระบบสมการเชิงอนุพันธ์สามัญ และใช้วิธีรุงเงอ-คุททาอันดับสี่หาผลเฉลยของระบบสมการดังกล่าว

ผลลัพธ์ที่ได้แสดงให้เห็นว่ามีความเป็นไปได้ในการเชื่อมสองคลื่นเชิงเดี่ยวที่ถูกวางนัยทั่วไปผ่านคลื่นกระแทก

สาขาวิชาคณิตศาสตร์ ลายมือชื่อนักศึกษา วิภารัตน์ วรพิทย์พงศ์

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GAS DYNAMICS EQUATION/RIEMANN WAVE/SIMPLE WAVE/SHOCK WAVE

It is known that the problem of a collision of two Riemann waves cannot be solved in the framework of using only constant solutions and Riemann waves. It happens that after collision of a shock wave and a rarefaction wave the entropy becomes non-constant, whereas entropy is constant in a Riemann wave. Recently, a new class of solutions of the gas dynamics equations was obtained. This class of solutions is called a class of generalized simple waves. The main feature of these solutions is that they are not isentropic, whereas their other properties are similar to the simple waves. In the present work, we consider two generalized simple waves separated by a shock wave.

Construction of a solution is split into two steps. For solving the problem of interaction of two generalized simple waves through a shock wave, we developed a code in MATLAB. Using REDUCE, the functions for the right-hand side of the system of ordinary differential equations are obtained. For solving this system, we applied the fourth-order Runge-Kutta method.

The obtained results show the possibility of joining two generalized simple waves through a shock wave.

School of Mathematics	Student's Signature ____
Academic Year 2015	Advisor's Signature ____

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Wiparat Worapitpong

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CHAPTER III

INTRODUCTION

The study of waves in gases is an important branch of science and engineering. One type of waves, well-known in the theory of the gas dynamics equations, are simple waves, which are also called Riemann waves (Courant and Friedrichs, 1990; Rozdestvenskii and Janenko, 1983). Riemann waves are widely exploited in the gas dynamics theory. The following two classical problems in this theory have an exact solution.

The first problem is the motion caused by a piston starting from rest and suddenly moving with constant velocity into a quiet gas. The motion of the piston

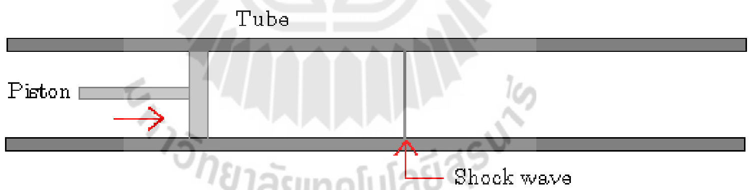

Figure 3.1 Piston moving into a quiet gas.

implies an immediate shock front, moving away from the piston with constant gas variables. Another problem is the problem of the motion caused by a piston moving with constant velocity out of a quiet gas. This motion of the piston develops a rarefaction simple wave (Riemann wave). The interaction of these two waves cannot be solved in the framework of using only constant solutions and Riemann waves. It happens that after collision of a shock wave and a rarefaction wave the entropy becomes non-constant, whereas entropy is constant in a Riemann wave.

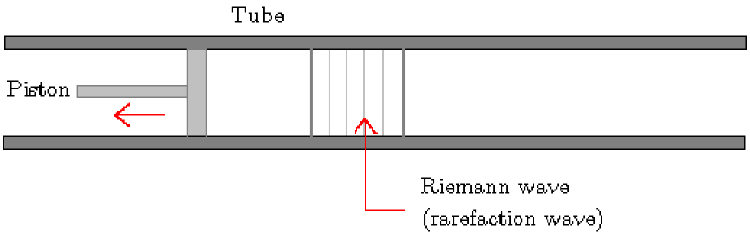

Figure 3.2 Piston moving out a quiet gas.

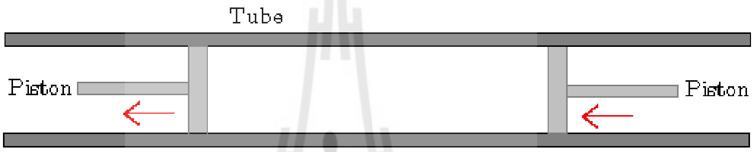

Figure 3.3 Combination of moving pistons.

The problem of a collision of a shock wave and a rarefaction (or simple) wave was studied for approximation of the gas dynamics equations, by excluding the energy equation from the study and considering that the pressure is a linear function of the density (Rozdestvenskii and Janenko, 1983).

Recently (Meleshko and Shapeev, 2011) a new class of solutions of the gas dynamics equations was obtained. This class of solutions is called the class of generalized simple waves. The main feature of these solutions is that they are not isentropic, whereas their other properties are similar to the simple waves. In this thesis the problem of interaction of two generalized simple waves through a shock wave is considered. In particular, if one of the generalized simple waves is reduced to a motion with a constant Riemann invariant, this study gives a solution of the classical problem of collision of a simple wave with a shock wave.

3.1 The objective

The main purpose of this thesis is to show the possibility of joining two generalized simple waves through a shock wave.

3.2 Organization

This thesis is organized as follows. In Chapter II, we introduce the gas dynamics equations and Rankine-Hugoniot conditions. In Chapter III, the Riemann waves, the generalized simple waves and equations along a characteristic are given. In Chapter IV the details of the method for solving the problem are shown. In Chapter V the conclusion is presented.

CHAPTER IV

GAS DYNAMICS EQUATIONS

This chapter provides some terminology and background of the theory of the gas dynamics equations.

Behavior of a gas is described by three conservation laws: conservation of mass, momentum and energy. Apart from conservation laws, thermodynamics of a gas has to be also introduced.

4.1 Conservation laws

In a moving continuous medium, for any moving volume $\omega_t \subset \Omega_t$ and any moment of time t the following equalities are satisfied:

$$\frac{d}{dt}(\int_{\omega_t} \rho \, d\omega) = 0,$$

$$\frac{d}{dt}(\int_{\omega_t} \rho \mathbf{v} \, d\omega) = \int_{\partial\omega_t} p_n \, d\sigma + \int_{\omega_t} \rho f \, d\omega,$$

$$\frac{d}{dt}(\int_{\omega_t} \rho (x \times \mathbf{v}) \, d\omega) = \int_{\partial\omega_t} (x \times p_n) \, d\sigma + \int_{\omega_t} \rho (x \times f) \, d\omega,$$

$$\frac{d}{dt}(\int_{\omega_t} \rho (\frac{\mathbf{v}^2}{2} + e) \, d\omega) = \int_{\partial\omega_t} \mathbf{v} p_n \, d\sigma + \int_{\omega_t} \rho \mathbf{v} f \, d\omega + \int_{\partial\omega_t} q_n \, d\sigma.$$

These equalities are called the conservation law of mass, the conservation law of linear momentum, the conservation law of angular momentum and the conservation law of energy, respectively,.

Here ω_t is material volume, $\partial\omega_t$ is the volume surface, ρ is the density per unit volume, $\mathbf{v}$ is the velocity, p_n is the density of internal surface force, q_n is the surface density of heat output, f is the body (external) force per unit mass, e is the internal energy per unit, x is a vector in three-dimensional space.

4.2 Gas dynamics axioms

The following axioms separate a gas from general continuum.

Axiom 4.1 (**Stress and heat**). For inviscid gas, the vectors p_n and q_n are defined as follows

$$p_n = -p\,n, \qquad q_n = 0,$$

where p is called the pressure and n is a unit vector.

Axiom 4.2 (**Thermodynamical behavior**). A gas is a reversible continuum, satisfying the thermodynamical law

$$\theta dS = de - \frac{p}{\rho^2}\,d\rho, \tag{4.1}$$

where θ is the temperature and S is the entropy.

The gas dynamics equations describe a motion of a two parametric continuum. The thermodynamic parameters ρ, p, e, the entropy S, and the temperature θ are related by the main thermodynamics identity (4.1).

In particular, choosing the thermodynamic parameters ρ and p, one assumes that $e = e(\rho, p)$, $\theta = \theta(\rho, p)$ and $S = S(\rho, p)$, where the functions $e(\rho, p)$, $\theta(\rho, p)$, $S(\rho, p)$ are related by the equations

$$\theta S_\rho = e_\rho - \frac{p}{\rho^2},$$

$$\theta S_p = e_p.$$

The thermodynamic equations are closed by the Clausius-Clapeyron relation

$$p = R\rho\theta, \tag{4.2}$$

where R is the specific gas constant. Notice that because of the Clausius-Clapeyron relation and the main thermodynamics identity one has that

$$e = E(\frac{p}{\rho}).$$

In particular, for a polytropic gas

$$e = \frac{1}{\gamma - 1}\ \frac{p}{\rho}, \tag{4.3}$$

where γ is the polytropic exponent.

4.2.1 Continuous motion

The integral conservation laws for a gas become

$$\begin{aligned} \frac{d}{dt}(\int_{\omega_t} \rho\, d\omega) &= 0, \\ \frac{d}{dt}(\int_{\omega_t} \rho\mathbf{v}\, d\omega) &= \int_{\partial\omega_t} -p \cdot n\, d\sigma, \\ \frac{d}{dt}(\int_{\omega_t} \rho(\frac{\mathbf{v}^2}{2} + e)\, d\omega) &= \int_{\partial\omega_t} \mathbf{v}(-p \cdot n)\, d\sigma. \end{aligned} \tag{4.4}$$

Remark 4.1. The conservation law of angular momentum is omitted because of the symmetry of the stress tensor $P = -pI$.

Definition 4.1 (Continuous motion). A motion of a continuum is called continuous in a domain Ω if the functions $\rho, e, \mathbf{v}, p$, and f are continuously differentiable functions in $\Omega \subset R^4$.

Because of arbitrariness of ω_t for a continuous motion, in the absence of a body force ($f = 0$) the integral equations (4.4) can be reduced to the following differential equations

$$\begin{aligned} \frac{d\rho}{dt} + \rho div(\mathbf{v}) &= 0, \\ \rho\ \frac{d\mathbf{v}}{dt} + \bigtriangledown p &= 0, \\ \rho\ \frac{de}{dt} + p div(\mathbf{v}) &= 0. \end{aligned} \tag{4.5}$$

In this thesis we study one-dimensional flow of a gas. In the one-dimensional flow all functions depend on (x,t), where $x \in R^1$ and the velocity is $\mathbf{v} = (v,0,0)$

In the one-dimensional case equations (4.5) become

$$\begin{aligned} \frac{\partial}{\partial t}\rho + \frac{\partial}{\partial x}(\rho v) = 0, \\ \frac{\partial}{\partial t}(\rho v) + \frac{\partial}{\partial x}(p+\rho v^2) = 0, \\ \frac{\partial}{\partial t}(\rho(e+\frac{v^2}{2})) + \frac{\partial}{\partial x}(v(p+\rho(e+\frac{v^2}{2}))) = 0. \end{aligned} \tag{4.6}$$

4.3 Shock conditions

In this project, discontinuous solutions of the gas dynamics equations are considered. In this case the notion of a solution is extended. A mathematical model of such solutions is constructed on the base of the integral conservation laws (4.4). The solution is considered as a piecewise smooth function. The requirement to satisfy the integral relations leads to the relations between the gas dynamics variables along the discontinuities. Assuming that a discontinuity occurs along the curve $x = X(t)$, the integral conservation laws lead to the equations (Chorin and Marsden, 1990; Courant and Friedrichs, 1948; Meleshko and Shapeev, 2011),

$$\begin{aligned} D[\ \rho\ ] = [\ \rho v\ ], \\ D[\ \rho v\ ] = [\ \rho v^2 + p\ ], \\ D\Big[\ \rho\Big(e+\frac{v^2}{2}\Big)\Big] = \Big[\ v\Big(p+\rho\Big(e+\frac{v^2}{2}\Big)\Big)\Big], \end{aligned} \tag{4.7}$$

where $D = X'(t)$ is the velocity of the moving discontinuity. The notation $[\ \cdot\ ]$ is used for a jump across the discontinuity, and it is defined as $[\ f\ ] = f_2$ - f_1 , with

$$f_1 = \lim_{\substack{(y,t)\to(x(t),t)\\(y,t)\in\Omega_1}} f(y-\tau),$$

$$f_2 = \lim_{\substack{(y,t)\to(x(t),t)\\(y,t)\in\Omega_2}} f(y+\tau),$$

where $\tau = 1/\rho > 0$, $\Omega = \Omega_1 \cup \Omega_2$ and f_i $(i = 1, 2)$ are the values of a function f from different sides of the discontinuity. These relations are called Rankine-Hugoniot conditions. Equations (4.7) can be simplified. In particular, for a polytropic gas they are

$$\begin{aligned} [\ \rho(v-D)\ ] &= 0, \\ \left[\ p + \rho(v-D)^2\ \right] &= 0, \\ \left[\ \frac{\gamma}{\gamma-1}\,\frac{p}{\rho} + \frac{(v-D)^2}{2}\ \right] &= 0. \end{aligned} \tag{4.8}$$

There is another representation of the Rankine-Hugoniot conditions, found by using the REDUCE program:

$$\begin{aligned} v_2 &= \frac{\mu(\rho_2^{-1} - \rho_1^{-1})}{(\gamma\rho_1 - \gamma\rho_2 + \rho_1 + \rho_2)} + v_1, \\ p_2 &= \frac{p_1(-\gamma\rho_1 + \gamma\rho_2 + \rho_1 + \rho_2)}{(\gamma\rho_1 - \gamma\rho_2 + \rho_1 + \rho_2)}, \\ D &= \frac{\mu\rho_1^{-1}}{(\gamma\rho_1 - \gamma\rho_2 + \rho_1 + \rho_2)} + v_1, \end{aligned} \tag{4.9}$$

where

$$\mu = -\sqrt{2\gamma p_1\rho_1\rho_2(\gamma\rho_1 - \gamma\rho_2 + \rho_1 + \rho_2)}.$$

CHAPTER V

GENERALIZED SIMPLE WAVES

This chapter introduces Riemann waves, generalized simple waves and relation of equations along a characteristic on generalized simple waves.

5.1 Riemann waves

Riemann waves are one of the well-known classes of solutions of the gas dynamics equations (Courant and Friedrichs, 1948; Rozdestvenskii and Janenko, 1983). These solutions can be obtained as follows. Assume that all the gas dynamics variables are functions of one dependent variable, say ρ:

$$v = V(\rho), \qquad p = P(\rho).$$

Substituting them into the gas dynamics equations, we obtain an overdetermined system of partial differential equations which in the matrix form is

$$\mathbf{A}\begin{pmatrix} \rho_t \\ \rho_x \end{pmatrix} = 0,$$

where

$$\mathbf{A} = \begin{pmatrix} 1 & v + \rho v_\rho \\ \rho v_\rho & \rho v v_\rho + P' \\ P' & vP' + \gamma P v_\rho \end{pmatrix}.$$

Since we try to find solutions such that ρ is not constant, then $rank\ \mathbf{A} < 2$.

The latter condition leads to the equations

$$\gamma P - \rho(\rho v_\rho)^2 = 0,$$

$$\rho P' = \gamma P.$$

Solving these equations (in Appendix B), we obtain

$$\begin{aligned} v + \frac{2\alpha}{\gamma - 1} &= c_1, \\ \frac{P}{\rho^\gamma} &= c_2, \end{aligned} \tag{5.1}$$

where $\alpha = \pm\sqrt{\frac{\gamma P}{\rho}}$ and c_1 and c_2 are constant. Solutions satisfying conditions (5.1) are called Riemann waves (or simple waves). Then the last equation of (5.1) allows us to conclude that the entropy S is constant in a Riemann wave. Notice that relations (5.1) can be also obtained from the conditions

$$\begin{aligned} \left(v + \frac{2\alpha}{\gamma - 1}\right)_x &= 0, \\ \left(\frac{p}{\rho^\gamma}\right)_x &= 0. \end{aligned} \tag{5.2}$$

The latter conditions (5.2) can be considered as an alternative definition of simple waves.

5.2 Generalized simple waves

A generalized simple wave is a solution of the gas dynamics equations (Meleshko and Shapeev, 2011) satisfying

$$\begin{aligned} \left(v + \frac{2\alpha}{\gamma - 1}\right)_x &= \phi, \\ \left(\frac{p}{\rho^\gamma}\right)_x &= \psi, \end{aligned} \tag{5.3}$$

where ϕ and ψ are some functions of the variables ρ, v and p. These functions are obtained from compatibility analysis of the overdetermined system of the partial differential equations consisting of the gas dynamics equations and equations (5.3). The compatibility conditions lead to (Meleshko and Shapeev, 2011)

$$\begin{gathered}\psi = k\rho^{\beta_1}p^{\beta}, \qquad \phi = -\tfrac{3\gamma}{(3\gamma-1)\alpha\rho}\psi, \\ \beta_1 = 1 - \tfrac{\gamma}{3\gamma-1}, \qquad \beta = 1 + \tfrac{1}{3\gamma-1}, \qquad k \neq 0, \qquad \alpha^2 = c^2.\end{gathered} \tag{5.4}$$

Solutions of the gas dynamics equations satisfying the conditions (5.3) are called generalized simple waves. In this thesis, this type of solutions is applied.

The equations determining a generalized simple wave can be rewritten in the form solved with respect to the main derivatives. In fact, the differential constraints (5.3) can be rewritten in the form

$$\begin{aligned} p_x &= \psi + \alpha^2\rho_x, \\ v_x &= \phi + \frac{\alpha}{\rho}\rho_x. \end{aligned} \tag{5.5}$$

Substituting (5.5) into the gas dynamics equation (4.6) they become

$$\begin{aligned} \rho_t &= -(v+\alpha)\rho_x - \rho\phi, \\ v_t &= -\frac{\alpha}{\rho}(v+\alpha)\rho_x - \frac{\psi}{\rho} - v\phi, \\ p_t &= -\alpha^2(v+\alpha)\rho_x - v\psi - \gamma p\phi. \end{aligned} \tag{5.6}$$

5.2.1 Equations along a characteristic

Consider a characteristic curve $x = x(t)$ on a generalized simple waves. The characteristic is determined by the equation

$$\frac{d}{dt}x(x(t),t) = v(x(t),t) + \alpha(x(t),t). \tag{5.7}$$

Then the derivatives of the gas dynamics variables are

$$
\begin{aligned}
\frac{d}{dt}\rho(x(t),t) &= \rho_x(x(t),t)\Big(v(x(t),t)+\alpha(x(t),t)\Big)+\rho_t(x(t),t),\\
\frac{d}{dt}v(x(t),t) &= v_x(x(t),t)\Big(v(x(t),t)+\alpha(x(t),t)\Big)+v_t(x(t),t),\\
\frac{d}{dt}p(x(t),t) &= p_x(x(t),t)\Big(v(x(t),t)+\alpha(x(t),t)\Big)+p_t(x(t),t).
\end{aligned}
\tag{5.8}
$$

Substituting (5.4), (5.5) and (5.6) into the latter equations, we obtain

$$
\begin{aligned}
\frac{d}{dt}x &= v+\alpha,\\
\frac{d}{dt}\rho &= \frac{3\gamma\psi}{(3\gamma-1)\alpha},\\
\frac{d}{dt}v &= \frac{\psi}{(3\gamma-1)\rho},\\
\frac{d}{dt}p &= \frac{\alpha\psi}{3\gamma-1}.
\end{aligned}
\tag{5.9}
$$

CHAPTER VI

STATEMENT OF THE PROBLEM AND THE SOLUTION METHOD

The method of solving the main problem of the thesis consists of two steps. In the first step, a shock wave and data on the shock wave are obtained. In the second step, using the obtained data as the initial data, the systems along characteristics are solved on both sides of the shock wave. Details of this study are presented in this Chapter.

6.1 Formulation of the problem

Consider a domain $\Omega \subset R^2$ separated by a curve $x = X(t) : \Omega = \Omega_1 \cup \Omega_2$. Assume that the curve $x = X(t)$ is a shock wave and solutions in Ω_1 and Ω_2 are generalized simple waves.

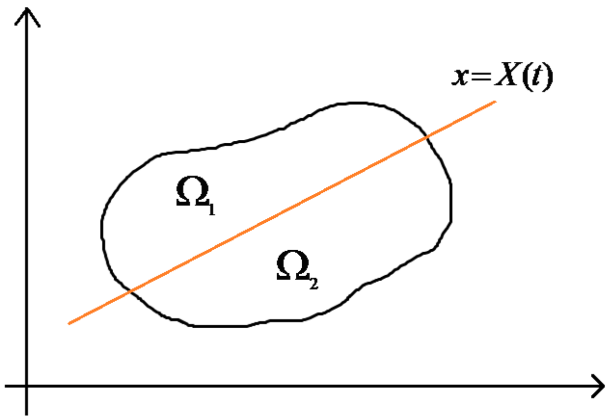

Figure 6.1 The curve of shock wave.

The main purpose of the thesis is to demonstrate that this solution can be realized.

6.2 Construction of equations along a shock wave

Consider any function $f(x,t)$ along the curve $x = X(t)$:

$$\varphi(t) = f(X(t), t).$$

Differentiating it with respect to t, one obtains

$$\varphi'(t) = f_t(X(t), t) + f_x(X(t), t)X'(t).$$

For a shock wave $x = X(t)$, one has

$$X'(t) = D,$$

where D is the velocity of the moving discontinuity.

As in the domains Ω_1 and Ω_2 solutions of the gas dynamics equations are continuously differentiable, one can write

$$\begin{aligned} \frac{d}{dt}\rho_i &= \rho_{it} + D\ \rho_{ix}, \\ \frac{d}{dt}v_i &= v_{it} + D\ v_{ix}, \\ \frac{d}{dt}p_i &= p_{it} + D\ p_{ix}, \end{aligned} \tag{6.1}$$

where $i = 1, 2$.

Differentiating the Rankine-Hugoniot conditions

$$\begin{aligned} \rho_2(v_2 - D) - \rho_1(v_1 - D) &= 0, \\ p_2 - p_1 + \rho_2(v_2 - D)^2 - \rho_1(v_1 - D)^2 &= 0, \\ \frac{\gamma}{\gamma - 1}\left(\frac{p_2}{\rho_2} - \frac{p_1}{\rho_1}\right) + \frac{1}{2}\left((v_2 - D)^2 - (v_1 - D)^2\right) &= 0, \end{aligned}$$

one derives

$$\frac{d\rho_2}{dt}(v_2 - D) + \rho_2\left(\frac{dv_2}{dt} - \frac{dD}{dt}\right) - \frac{d\rho_1}{dt}(v_1 - D) - \rho_1\left(\frac{dv_1}{dt} - \frac{dD}{dt}\right) = 0,$$

$$\begin{aligned}\frac{dp_2}{dt} - \frac{dp_1}{dt} + \frac{d\rho_2}{dt}(v_2 - D)^2 + 2\rho_2(v_2 - D)\Big(\frac{dv_2}{dt} - \frac{dD}{dt}\Big)\\ -\frac{d\rho_1}{dt}(v_1 - D)^2 - 2\rho_1(v_1 - D)\Big(\frac{dv_1}{dt} - \frac{dD}{dt}\Big) = 0, \quad (6.2)\end{aligned}$$

$$\begin{aligned}\frac{\gamma}{(\gamma-1){\rho_2}^2}\Big(\rho_2\frac{dp_2}{dt} - p_2\frac{d\rho_2}{dt}\Big) - \frac{\gamma}{(\gamma-1){\rho_1}^2}\Big(\rho_1\frac{dp_1}{dt} - p_1\frac{d\rho_1}{dt}\Big)\\ +(v_2 - D)\Big(\frac{dv_2}{dt} - \frac{dD}{dt}\Big) - (v_1 - D)\Big(\frac{dv_1}{dt} - \frac{dD}{dt}\Big) = 0.\end{aligned}$$

Substituting (6.1) into these equations and then into (5.6), we derive an algebraic system with respect to ρ_{1x}, ρ_{2x} and $\frac{dD}{dt}$. For the sake of simplicity we demonstrate here this derivation using the first equation of the Rankine-Hugoniot conditions

$$\rho_2(v_2 - D) - \rho_1(v_1 - D) = 0.$$

Differentiating with respect to t,

$$\rho_2\left(\frac{dv_2}{dt} - \frac{dD}{dt}\right) + \frac{d\rho_2}{dt}(v_2 - D) = \rho_1\left(\frac{dv_1}{dt} - \frac{dD}{dt}\right) + \frac{d\rho_1}{dt}(v_1 - D). \quad (6.3)$$

Substituting (6.1), we have

$$\begin{aligned}&\rho_2(v_{2t} + Dv_{2x}) - \rho_2\frac{dD}{dt} + (v_2 - D)(\rho_{2t} + D\rho_{2x})\\ &\qquad = \rho_1(v_{1t} + Dv_{1x}) - \rho_1\frac{dD}{dt} + (v_1 - D)(\rho_{1t} + D\rho_{1x}),\end{aligned}$$

$$\begin{aligned}&\rho_2 v_{2t} + D\rho_2 v_{2x} - \rho_2\frac{dD}{dt} + (v_2 - D)\rho_{2t} + (v_2 - D)D\rho_{2x}\\ &\qquad = \rho_1 v_{1t} + D\rho_1 v_{1x} - \rho_1\frac{dD}{dt} + (v_1 - D)\rho_{1t} + (v_1 - D)D\rho_{1x}.\end{aligned}$$

Substituting (5.5) and (5.6), the left-hand side becomes

$$\rho_2\Big(-\frac{\alpha_2}{\rho_2}(v_2+\alpha_2)\rho_{2x}-\frac{\psi_2}{\rho_2}-v_2\phi_2\Big)+D\rho_2\Big(\phi_2+\frac{\alpha_2}{\rho_2}\rho_{2x}\Big)$$
$$+(v_2-D)\Big(-(v_2+\alpha_2)\rho_{2x}-\rho_2\phi_2\Big)+(v_2-D)D\rho_{2x}-\rho_2\frac{dD}{dt}$$

$$=-\alpha_2(v_2+\alpha_2)\rho_{2x}-\psi_2-\rho_2v_2\phi_2+D\rho_2\phi_2+D\alpha_2\rho_{2x}-(v_2-D)(v_2+\alpha_2)\rho_{2x}$$
$$-(v_2-D)\rho_2\phi_2+(v_2-D)D\rho_{2x}-\rho_2\frac{dD}{dt}$$

$$=\Big(-\alpha_2(v_2+\alpha_2)+D\alpha_2-(v_2-D)(v_2+\alpha_2)+(v_2-D)D\Big)\rho_{2x}$$
$$-\psi_2-\rho_2v_2\phi_2+D\rho_2\phi_2-(v_2-D)\rho_2\phi_2-\rho_2\frac{dD}{dt}$$

$$=-(D-v_2+\alpha_2)^2\rho_{2x}-\rho_2\frac{dD}{dt}-\psi_2+(D-v_2)\rho_2\phi_2-(v_2-D)\rho_2\phi_2$$

$$=-(D-v_2+\alpha_2)^2\rho_{2x}-\rho_2\frac{dD}{dt}-\psi_2+2(D-v_2)\rho_2\phi_2.$$

The right-hand side becomes

$$-(D-v_1+\alpha_1)^2\rho_{1x}-\rho_1\frac{dD}{dt}-\psi_1+2(D-v_1)\rho_1\phi_1.$$

Thus we obtain

$$(D-v_1+\alpha_1)^2\rho_{1x}-(D-v_2+\alpha_2)^2\rho_{2x}+(\rho_2-\rho_1)\frac{dD}{dt}=$$
$$\psi_1-\psi_2+2(D-v_2)\rho_2\phi_2-2(D-v_1)\rho_1\phi_1. \qquad (6.4)$$

Applying a similar method to the study of the other equations of the Rankine-Hugoniot conditions we obtain the following system of equations,

$$(D - v_1 + \alpha_1)^2\rho_{1x} - (D - v_2 + \alpha_2)^2\rho_{2x} + (\rho_2 - \rho_1)\frac{dD}{dt} =$$
$$\psi_1 - \psi_2 + 2(D - v_2)\rho_2\phi_2 - 2(D - v_1)\rho_1\phi_1$$

$$\begin{aligned}
-(D - v_1 + \alpha_1)^3\rho_{1x} + (D - v_2 + \alpha_2)^3\rho_{2x} - 2(\rho_2 v_2 - \rho_1 v_1 + D(\rho_1 - \rho_2)\frac{dD}{dt} \\
= 3\rho_2\phi_2(D - v_2)^2 - 3\rho_1\phi_1(D - v_1)^2 + \gamma(p_2\phi_2 - p_1\phi_1) \\
+ 3\psi_1(D - v_1) - 3\psi_2(D - v_2)
\end{aligned} \tag{6.5}$$

$$\begin{aligned}
\frac{\alpha_1}{\rho_1}(D - v_1 + \alpha_1)^2\rho_{1x} - \frac{\alpha_2}{\rho_2}(D - v_2 + \alpha_2)^2\rho_{2x} + (v_1 - v_2)\frac{dD}{dt} \\
= \frac{2\gamma - 1}{\gamma - 1}\Big((D - v_1)\frac{\psi_1}{\rho_1} - (D - v_2)\frac{\psi_2}{\rho_2}\Big) \\
+ \phi_2\Big({\alpha_2}^2 + (D - v_2)^2\Big) - \phi_1\Big({\alpha_1}^2 + (D - v_1)^2\Big)
\end{aligned}$$

This system is an algebraic system of equations with respect to ρ_{1x}, ρ_{2x} and $\frac{dD}{dt}$:

$$h_{i1}\rho_{1x} + h_{i2}\rho_{2x} + h_{i3}\frac{dD}{dt} = h_{i4}, \quad (i = 1, 2, 3), \tag{6.6}$$

where

$$h_{11} = (D - v_1 + \alpha_1)^2,$$

$$h_{12} = -(D - v_2 + \alpha_2)^2,$$

$$h_{13} = \rho_2 - \rho_1,$$

$$h_{14} = \psi_1 - \psi_2 + 2(D - v_2)\rho_2\phi_2 - 2(D - v_1)\rho_1\phi_1,$$

$$h_{21} = -(D - v_1 + \alpha_1)^3,$$

$$h_{22} = (D - v_2 + \alpha_2)^3,$$

$$h_{23} = 0,$$

$$h_{24} = 3\rho_2\phi_2(D-v_2)^2 - 3\rho_1\phi_1(D-v_1)^2 + \gamma(p_2\phi_2 - p_1\phi_1)$$
$$+ 3\psi_1(D-v_1) - 3\psi_2(D-v_2),$$
$$h_{31} = \frac{\alpha_1}{\rho_1}(D - v_1 + \alpha_1)^2,$$
$$h_{32} = -\frac{\alpha_2}{\rho_2}(D - v_2 + \alpha_2)^2,$$
$$h_{33} = v_1 - v_2,$$
$$h_{34} = \frac{2\gamma-1}{\gamma-1}\Big((D-v_1)\frac{\psi_1}{\rho_1} - (D-v_2)\frac{\psi_2}{\rho_2}\Big)$$
$$+ \phi_2\Big({\alpha_2}^2 + (D-v_2)^2\Big) - \phi_1\Big({\alpha_1}^2 + (D-v_1)^2\Big).$$

In matrix form, this system is

$$\begin{pmatrix} h_{11} & h_{12} & h_{13} \\ h_{21} & h_{22} & h_{23} \\ h_{31} & h_{32} & h_{33} \end{pmatrix} \cdot \begin{pmatrix} \rho_{1x} \\ \rho_{2x} \\ \frac{dD}{dt} \end{pmatrix} = \begin{pmatrix} h_{14} \\ h_{24} \\ h_{34} \end{pmatrix}.$$

The determinant of this system is

$$\triangle = (\alpha_1 + D - v_1)^2(\alpha_2 + D - v_2)^2\Big(\alpha_1\alpha_2(\rho_1 - \rho_2)^2 - \rho_1\rho_2(v_1 - v_2)^2\Big)/(\rho_1\rho_2)$$

Since $D \neq v_i - \alpha_i,\ (i = 1, 2)$, then the determinant of this system is not equal to zero if

$$\frac{\alpha_1\alpha_2}{\rho_1\rho_2} \neq \left(\frac{v_1 - v_2}{\rho_1 - \rho_2}\right)^2. \tag{6.7}$$

Substituting $D,\ p_2$ and v_2 from (4.9) into this system, and applying the Gauss elimination method, we find $\rho_{1x},\ \rho_{2x}$, $\dot{\mathrm{D}}$, where $\dot{\mathrm{D}} := \frac{dD}{dt}$. Their expressions are shown in Appendix A.

Substituting $\rho_{1x},\ \rho_{2x}$, $\dot{\mathrm{D}}$ into (6.1), we obtain the system of ordinary differential equations

$$\begin{aligned}\frac{d}{dt}x\ &= D,\\ \frac{d}{dt}\rho_1 &= f_1(\rho_1,\rho_2,v_1,p_1),\\ \frac{d}{dt}v_1 &= f_2(\rho_1,\rho_2,v_1,p_1),\\ \frac{d}{dt}p_1 &= f_3(\rho_1,\rho_2,v_1,p_1),\\ \frac{d}{dt}\rho_2 &= f_4(\rho_1,\rho_2,v_1,p_1),\end{aligned} \tag{6.8}$$

where for the sake of shortness, the representations of the functions $f_i(i = 1,2,3,4,5)$ are presented in Appendix A. Here other gas dynamic variables on the shock wave are found from the Rankine-Hugoniot conditions (4.9).

6.3 Algorithm for finding data on a shock wave

Consider the initial data at $t = 0$:

$$x\ = x_0,\quad \rho_1 = \rho_{10},\quad \rho_2 = \rho_{20},\quad v_1 = v_{10},\quad p_1 = p_{10}, \tag{6.9}$$

such that conditions (6.7) are satisfied.

The algorithm of finding the shock wave and the data on it consists of the following steps.

First, one needs to determine the right-hand side in equations (6.8). The code of this step is given in file "w.m":

1. Set the initial data $x_0,\ \rho_{10},\ \rho_{20}$, and v_{10}.
2. Using (4.9), find $v_2,\ p_2$, and D : (lines 12-24, in "w.m").
3. Applying the Gauss elimination method, find $\rho_{1x},\ \rho_{2x}$, and $\dot{\mathrm{D}}$: (lines 32-79, in "w.m").

4. Compute the right-hand side of ordinary differential equations (6.8) : (lines 81-103, in “w.m”).

5. The data on the shock wave are obtained by the fourth-order Runge-Kutta method *. The code of this part is given in file “rk_w.m” (lines 1-23).

In Figure (6.2) results of calculations of the shock wave are presented. For computations we used the following data:

$$k_1 \;=\; 0.01, \qquad k_2 \;=\; 0, \qquad \gamma \;=\; 1.4$$

and the initial data

$$x_0 \;=\; 1, \qquad \rho_1 \;=\; 1, \qquad \rho_2 \;=\; 2, \qquad v_1 \;=\; 2, \qquad p_1 \;=\; 2.$$

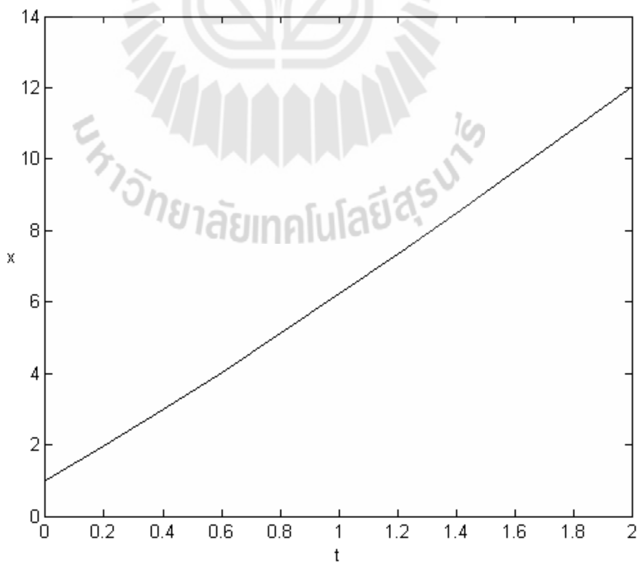

Figure 6.2 Shock Wave.

*Runge-Kutta method is presented in Appendix A

6.4 Adjoining two generalized simple waves

The solutions on both sides of the shock wave are obtained by integrating equations (6.8) along characteristics starting from a point on the shock wave.

The algorithm consists of the following steps.

1. Choose a point on the shock wave.
2. Integrate equations (5.9) using the fourth-order Runge-Kutta method with the initial conditions obtained on the shock wave at the chosen point : (line 25-44, in "rk_w.m").
3. Change the point and continue with item 1 : (line 45-70, in "rk_w.m").
4. Consider all points o the shock wave. All computed data are plotted on figure 6.3-6.5 : (line 71-97, in "rk_w.m").

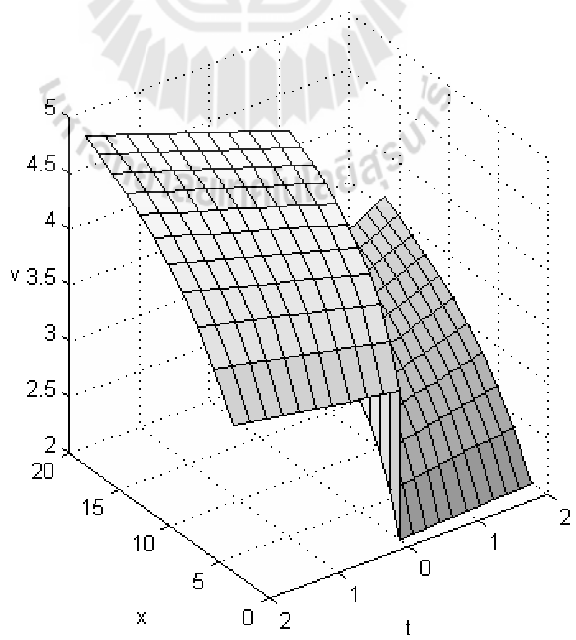

Figure 6.3 Velocity on (t, x, v) - space.

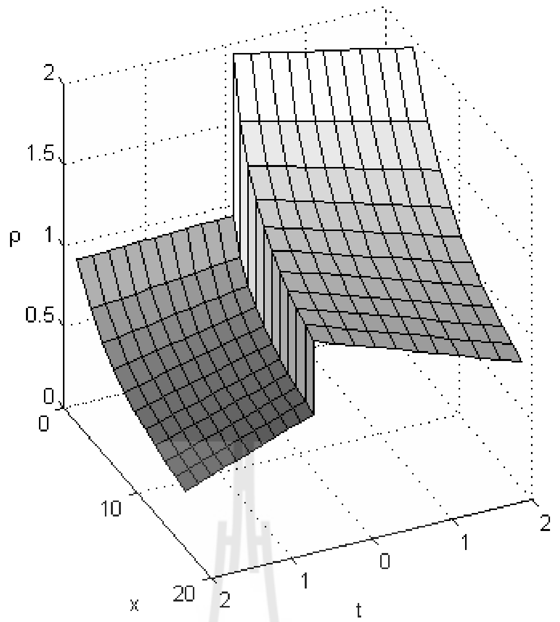

Figure 6.4 Density on (t, x, ρ) - space.

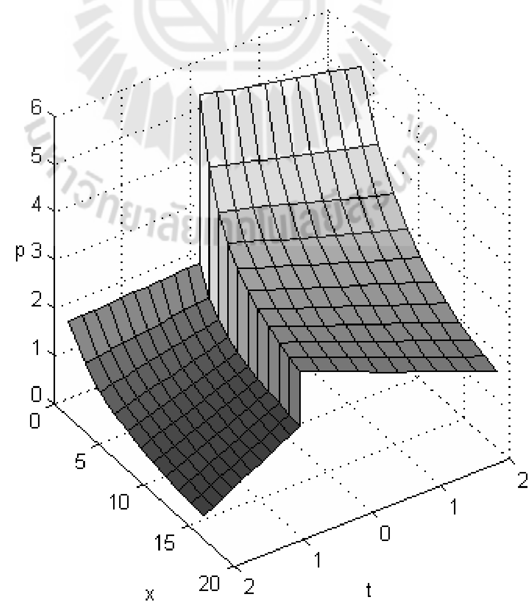

Figure 6.5 Pressure on (t, x, p) - space.

CHAPTER VII

CONCLUSIONS

The obtained results show the possibility of joining two generalized simple waves through a shock wave.

The solution of the problem depends on six constants which value of the functions $v_1,\ p_1,\ \rho_1$ and ρ_2 at some point t_0 and x_0, and two constants k_1 and k_2 characterizing the generalized simple waves. The solutions found are presented in Figures 4.2-4.5 prepared in MATLAB (version2011a).

REFERENCES

REFERENCES

Chorin, A. J. and Marsden, J. E. (1990). **A Mathematical Introduction to Fluid Mechanics.** Springer: New York.

Courant, R. and Friedrichs, K. O. (1948). **Supersonic Flow And Shock Waves.** Interscience Publishers.

Meleshko, S. V. (2005). **Methods for Constructing Exact Solutions of Partial Differential Equations: Mathematical and Analytical Techniques with Applications to Engineering.** Springer: New York.

Meleshko, S. V. and Shapeev, V. P. (2011). Nonisentropic solutions of simple wave type of the gas dynamics equations. **Journal of Nonlinear Mathematical Physics.** 18: 195-212.

Rozdestvenskii, B. L. and Janenko, N. N. (1983). **Systems of Quasilinear Equations And Their Applications to Gas Dynamics.** (English translation, Schulenberger, J. R., Translation of mathematical monographs, V.55, American Mathematical Society, Rhode Island.)

Sidorov, A. F., Shapeev, V. P. and Yanenko, N. N. (1984). **The Method of Differential Constraints and Its Applications in Gas Dynamics.** Nauka: Novosibirsk.

Yanenko, N. N. (1964). Compatibility theory and methods of integrating systems of nonlinear partial differential equations. **In: Proceedings of the Fourth All-Union Mathematics Congress.** 2: 613-621.

APPENDICES

APPENDIX A

COMPUTER PROGRAMS

This appendix contains codes of the programs which are written in MATLAB.

CODES FOR FINDING A SOLUTION ON THE SHOCK WAVE

File name w.m

```
%Author: Wiparat Worapitpong
%This program for find y:x, ro1, ro2, u1, p1
%Which get D, u2, p2 from REDUCE
%And set k1=0.01, k2=0
function dy=w(t,y)
syms ro1 ro2 u1 p1
ro1=y(2);
u1=y(3);
p1=y(4);
ro2=y(5);
%-----------------------------------------------------------
k1=0.01;            %constant from side 1
k2=0;               %constant from side 2
ga=1.4;             %Specific gas constant: Polytropic gas
be=1+(1/(3*ga-1));
be1=1-(ga/(3*ga-1));
%-----------------------------------------------------------
```

```
so2=2*ga*p1*ro1*ro2*(ga*ro1-ga*ro2+ro1+ro2);
sq=-sqrt(so2);
%------------------------------------------------------------
u2=sq*(-ro1^(-1)+ro2^(-1))/(ga*ro1-ga*ro2+ro1+ro2)+u1;
p2=(p1*(-ga*ro1+ga*ro2+ro1+ro2))/(ga*ro1-ga*ro2+ro1+ro2);
d=-(sq*ro1^(-1))/(ga*ro1-ga*ro2+ro1+ro2)+u1;
%------------------------------------------------------------
alf1=(ga*p1/ro1)^(1/2);
alf2=(ga*p2/ro2)^(1/2);
ps1=k1*(ro1^be1)*(p1^be);
fi1=-3*ga*ps1/((3*ga-1)*alf1*ro1);
ps2=k2*(ro2^be1)*(p2^be);
fi2=-3*ga*ps2/((3*ga-1)*alf2*ro2);
%------------------------------------------------------------
aa = sym(zeros(3));

aa(1,1)=-alf1^2+2*alf1*(-d+u1)-d^2+2*d*u1-u1^2;
aa(1,2)=alf2^2+2*alf2*(d-u2)+d^2-2*d*u2+u2^2;
aa(1,3)=-ro1+ro2;
aa(2,1)=alf1^3+3*alf1^2*(d-u1)+3*alf1*(d^2-2*d*u1+u1^2)+...
        d^3-3*d^2*u1+3*d*u1^2-u1^3;
aa(2,2)=-alf2^3+3*alf2^2*(-d+u2)+3*alf2*(-d^2+2*d*u2-u2^2)-...
        d^3+3*d^2*u2-3*d*u2^2+u2^3;
aa(2,3)=2*(d*ro1-d*ro2-ro1*u1+ro2*u2);
aa(3,1)=(alf1^3+2*alf1^2*(d-u1)+alf1*(d^2-2*d*u1+u1^2))/ro1;
aa(3,2)=(-alf2^3+2*alf2^2*(-d+u2)+alf2*(-d^2+2*d*u2-u2^2))/ro2;
```

```
aa(3,3)=-u1+u2;
%--------------------------------------------------------
aa_10=(ps1*alf2*(-3*alf1*ga+alf1-6*d*ga+6*ga*u1)+...
        ps2*alf1*(3*alf2*ga-alf2+6*d*ga-...
        6*ga*u2))/(alf1*alf2*(3*ga-1));

aa_20=(3*ps1*alf2*(alf1^2*ga+3*alf1*d*ga-alf1*d-3*alf1*ga*u1+...
        alf1*u1+3*d^2*ga-6*d*ga*u1+3*ga*u1^2)+...
        3*ps2*alf1*(-alf2^2*ga-3*alf2*d*ga+alf2*d+...
        3*alf2*ga*u2-alf2*u2-3*d^2*ga+6*d*ga*u2-...
        3*ga*u2^2))/(alf1*alf2*(3*ga-1));

aa_30=(ps1*alf2*ro2*(3*alf1^2*ga^2-3*alf1^2*ga+6*alf1*d*ga^2-...
        5*alf1*d*ga+alf1*d-6*alf1*ga^2*u1+5*alf1*ga*u1-...
        alf1*u1+3*d^2*ga^2-3*d^2*ga-6*d*ga^2*u1+6*d*ga*u1+...
        3*ga^2*u1^2-3*ga*u1^2)+ps2*alf1*ro1*(-3*alf2^2*ga^2+...
        3*alf2^2*ga-6*alf2*d*ga^2+5*alf2*d*ga-alf2*d+...
        6*alf2*ga^2*u2-5*alf2*ga*u2+alf2*u2-3*d^2*ga^2+...
        3*d^2*ga+6*d*ga^2*u2-6*d*ga*u2-3*ga^2*u2^2+...
        3*ga*u2^2))/(alf1*alf2*ro1*ro2*(3*ga^2-4*ga+1));
%--------------------------------------------------------
b = sym(zeros(3,1));
b(1,1)=-aa_10;
b(2,1)=-aa_20;
b(3,1)=-aa_30;
b=simplify(b);
```

```
b=collect(b);

aa1=simplify(aa);

m = sym(zeros(3,1));
m = aa1\b;

syms ro1_x  ro2_x  d_t
ro1_x = m(1,1);
ro2_x = m(2,1);
%-----------------------------------------------------------
dro1=(-((p1*ro1_x-3*ps1*ro1-3*ga*p1*ro1_x)*ga-...
    (d-u1)*(3*ga-1)*alf1*ro1*ro1_x))/(alf1*(3*ga-1)*ro1);

du1=(-((p1*ro1_x-3*ps1*ro1-3*ga*p1*ro1_x)*ga*u1+...
    (ga*p1*ro1_x+ps1*ro1)*(3*ga-1)*alf1-...
    (p1*ro1_x-3*ps1*ro1-3*ga*p1*ro1_x)*d*ga))/(alf1*(3*ga-...
    1)*ro1^2);

dp1=(-((p1*ro1_x-3*ps1*ro1-3*ga*p1*ro1_x)*alf1*ga-...
    (d-u1)*(ga*p1*ro1_x+ps1*ro1)*(3*ga-1)))/((3*ga-1)*ro1);

dro2=(-((p2*ro2_x-3*ps2*ro2-3*ga*p2*ro2_x)*ga-...
    (d-u2)*(3*ga-1)*alf2*ro2*ro2_x))/(alf2*(3*ga-1)*ro2);
%-----------------------------------------------------------

```

```
dy = zeros(5,1);
dy(1) = d;
dy(2) = dro1;
dy(3) = du1;
dy(4) = dp1;
dy(5) = dro2;
end
%----------------------------------------------------------
```

CONSTRUCTION OF A SOLUTION ON BOTH SIDES OF THE SHOCK WAVE

For side 1

File name wp1.m

```
%Author: Wiparat Worapitpong
%This program to find solution on 1st side.
%with initial point from w.m
function dy=wp1(t,y)
syms ro1 u1 p1
ro1=y(2);
u1=y(3);
p1=y(4);

k1=0.01;              %constant from side 1
ga=1.4;               %Specific gas constant: Polytropic gas
```

```
be=1+(1/(3*ga-1));
be1=1-(ga/(3*ga-1));

alf1=(ga*p1/ro1)^(1/2);
ps1=k1*(ro1^be1)*(p1^be);
fi1=-3*ga*ps1/((3*ga-1)*alf1*ro1);

dro1=(3*ga*fi1)/alf1*(3*ga-1);
du1=(fi1)/ro1*(3*ga-1);
dp1=(alf1*fi1)/(3*ga-1);

dy = zeros(4,1);
dy(1) = u1-alf1;
dy(2) = dro1;
dy(3) = du1;
dy(4) = dp1;
end
```

For side 2

File name wp2.m

```
%Author: Wiparat Worapitpong
%This program to find solution on 2st side.
%with initial point from w.m
function dy=wp2(t,y)
syms ro2 u2 p2
ro2=y(2);
```

```
u2=y(3);
p2=y(4);

k2=0.0;                 %constant from side 2
ga=1.4;                 %Specific gas constant: Polytropic gas
be=1+(1/(3*ga-1));
be1=1-(ga/(3*ga-1));

alf2=(ga*p2/ro2)^(1/2);
ps2=k2*(ro2^be1)*(p2^be);
fi2=-3*ga*ps2/((3*ga-1)*alf2*ro2);

dro2=(3*ga*fi2)/(alf2*(3*ga-1));
du2=(fi2)/(ro2*(3*ga-1));
dp2=(alf2*fi2)/(3*ga-1);

dy = zeros(4,1);
dy(1) = u2-alf2;
dy(2) = dro2;
dy(3) = du2;
dy(4) = dp2;
end
```

THE FOURTH-ORDER RUNGE-KUTTA METHOD

The fourth-order Runge-Kutta method is a numerical method for solving the Cauchy problem.

$$y'(t) \;=\; f(t, y(t)),$$
$$y(t_0) \;=\; y_0.$$

in from

$$y_{i+1} \;=\; y_i \;+\; \frac{h}{6}\Big[\;k_1 \;+2k_2 \;+2k_3 \;+k_4\Big],$$
$$t_{i+1} \;=\; t_i \;+\; h, \qquad (i \;=\; 0, 1, ..., N-1),$$

where

$$k_1 \;=\; f(t_i, y_i),$$
$$k_2 \;=\; f\Big(t_i \;+\; \frac{h}{2}, y_i \;+\; \frac{h}{2}\;k_1\Big),$$
$$k_3 \;=\; f\Big(t_i \;+\; \frac{h}{2}, y_i \;+\; \frac{h}{2}\;k_2\Big),$$
$$k_4 \;=\; f(t_i \;+\; h, y_i \;+\; h\;k_3).$$

CODES FOR FINDING SOLUTION ON THE SHOCK WAVE

For shock wave

File name rk_w.m

```
function [t,y]=rk_w(w,y0,h,a,b)
% y0 is initial data vector
% w is functions of equations.
% h is step length,
% [a, b] is interval
n=floor((b-a)/h);       % evaluate step length
```

```
t(1)=a;                 % take left point of interval, t is a vector
y(:,1)=y0;
% take initial data from y0, note that dimension of vector y
for i=1:n
t(i+1)=t(i)+h;
kk1=w(t(i),y(:,i));
kk2=w(t(i)+h/2,y(:,i)+h*kk1/2);
kk3=w(t(i)+h/2,y(:,i)+h*kk2/2);
kk4=w(t(i)+h,y(:,i)+h*kk3);
y(:,i+1)=y(:,i)+h*(kk1+2*kk2+2*kk3+kk4)/6;
end
%-----------------------------------------------------------
plot(t,y(1,:))
xlabel('t');
ylabel('x');
set(get(gca,'YLabel'),'Rotation',0.0);
%title('Solution on the Shock Wave');

% Keep result
A=transpose(y);
[m,n]=size(A);

% Step 2 : Compute solution on side 1
%--------------------- Code for side 1 --------------------
for i=1:m
    [t,y]=rk_wp1(@wp1,[A(i,1:4)],h,a,b);
```

```
    x(i,:)=y(1,:);   % Keep x
    ro(i,:)=y(2,:);
    u(i,:)=y(3,:);   % Keep u
    p(i,:)=y(4,:);
end
%--------------------- Code for side 2 --------------------
ro1=A(:,2);
u1=A(:,3);
p1=A(:,4);
ro2=A(:,5);
ga=1.4;
%----------------------------------------------------------
% Step 3 : Compute u2 and p2
for i=1:m
so2(i)=2*ga*p1(i)*ro1(i)*ro2(i)*(ga*ro1(i)-...
        ga*ro2(i)+ro1(i)+ro2(i));
sq(i)=-sqrt(so2(i));
u2(i)=sq(i)*(-ro1(i)^(-1)+ro2(i)^(-1))/(ga*ro1(i)-...
        ga*ro2(i)+ro1(i)+ro2(i))+u1(i);
p2(i)=(p1(i)*(-ga*ro1(i)+ga*ro2(i)+ro1(i)+...
        ro2(i)))/(ga*ro1(i)-ga*ro2(i)+ro1(i)+ro2(i));
end

% Step 4 : Compute solution on side 2
for i=1:m
    [t,y]=rk_wp2(@wp2,[A(i,1) ro2(i) u2(i) p2(i)],h,a,b);
```

```
    xx(i,:)=y(1,:);   % Keep x2 from side 2
    rr(i,:)=y(2,:);
    uu(i,:)=y(3,:);   % Keep u2 from side 2
    pp(i,:)=y(4,:);
end

% Step 5 : Change(flip) column xx and uu for plot graph
x_2=fliplr(xx);
r_2=fliplr(rr);
u_2=fliplr(uu);
p_2=fliplr(pp);

% Step 6 : Write in 1 array
tt=-t;
tt=fliplr(tt);

st=[tt t];
sx=[x_2 x];
su=[u_2 u];
sr=[r_2 ro];
sp=[p_2 p];

% Step 7 : Plot graph 3D

surf(st,sx,su);
xlabel('t');
```

```
ylabel('x');
zlabel('v');
%title('Adjoining of 2 Waves');

surf(st,sx,sr);
xlabel('t');
ylabel('x');
zlabel('\rho');

surf(st,sx,sp);
xlabel('t');
ylabel('x');
zlabel('p');
```

CODES FOR FINDING SOLUTION ON THE BOTH SIDES OF THE SHOCK WAVE

Find a solution on side 1 by Runge-Kutta fourth-order method

File name rk_wp1.m

```
% y0=[x0 ro1 u1 p1]
function [t,y]=rk_wp1(wp1,y0,h,a,b)
% y0 is initial data vector
% wp1 is functions of equations. i.e. subprogram,
% h is step length,
% a is left point of interval, b is right point of interval
n=floor((b-a)/h);      % evaluate step length
```

```
t(1)=a;                 % take left point of interval, t is a vector
y(:,1)=y0;
% take initial data from y0, note that dimension of vector y
for i=1:n
    t(i+1)=t(i)+h;
    kk1=wp1(t(i),y(:,i));
    kk2=wp1(t(i)+h/2,y(:,i)+h*kk1/2);
    kk3=wp1(t(i)+h/2,y(:,i)+h*kk2/2);
    kk4=wp1(t(i)+h,y(:,i)+h*kk3);
    y(:,i+1)=y(:,i)+h*(kk1+2*kk2+2*kk3+kk4)/6;
end
```

Find a solution on side 2 by Runge-Kutta fourth-order method

File name rk_wp2.m

```
% y0=[x0 ro2 u2 p2]
function [t,y]=rk_wp2(wp2,y0,h,a,b)
% y0 is initial data vector
% wp2 is functions of equations. i.e. subprogram,
% h is step length,
% a is left point of interval, b is right point of interval
n=floor((b-a)/h);       % evaluate step length
t(1)=a;                 % take left point of interval, t is a vector
y(:,1)=y0;
% take initial data from y0, note that dimension of vector y
for i=1:n
    t(i+1)=t(i)+h;
```

```
    kk1=wp2(t(i),y(:,i));
    kk2=wp2(t(i)+h/2,y(:,i)+h*kk1/2);
    kk3=wp2(t(i)+h/2,y(:,i)+h*kk2/2);
    kk4=wp2(t(i)+h,y(:,i)+h*kk3);
    y(:,i+1)=y(:,i)+h*(kk1+2*kk2+2*kk3+kk4)/6;
end
```

APPENDIX B

PROOFS

This appendix contains the proof of some parts from Chapter II and Chapter III.

B.1 Derivation of the Rankine-Hugoniot conditions for a polytropic gas

To show that the Rankine-Hugoniot conditions

$$\begin{aligned} D[\rho] &= [\rho v], \\ D[\rho v] &= [\rho v^2 + p], \\ D\Big[\rho(e + \frac{v^2}{2})\Big] &= \Big[v(p + \rho(e + \frac{v^2}{2}))\Big] \end{aligned} \tag{B.1}$$

for a polytropic gas are

$$\begin{aligned} [\rho(v - D)] &= 0, \\ \Big[p + \rho(v - D)^2\Big] &= 0, \\ \Big[\frac{\gamma}{\gamma - 1}\frac{p}{\rho} + \frac{(v - D)^2}{2}\Big] &= 0. \end{aligned} \tag{B.2}$$

Proof. Since the notation [$\cdot$] is defined as [f] = f_2 - f_1 , where f_1 and f_2 are values of a function f from different sides of the discontinuity. Then first equation of (B.1) can be rewritten

$$
\begin{aligned}
\rho_2 v_2 - \rho_1 v_1 &= D(\rho_2 - \rho_1), \\
\rho_2 v_2 - \rho_1 v_1 - D\rho_2 + D\rho_1 &= 0, \\
(\rho_2 v_2 - D\rho_2) - (\rho_1 v_1 - D\rho_1) &= 0, \\
[\rho(v - D)] &= 0.
\end{aligned} \tag{B.3}
$$

The second equation of (B.1) can be rewritten

$$
\begin{aligned}
[\rho v^2 + p] &= D[\rho v], \\
[\rho v^2] + [p] - D[\rho v] &= 0, \\
[p] + [\rho v^2 - D\rho v] &= 0, \\
[p] + [\rho v(v - D)] &= 0.
\end{aligned}
$$

Consider

$$
\begin{aligned}
[\rho v(v - D)] &= \rho_2 v_2(v_2 - D) - \rho_1 v_1(v_1 - D), \\
&= \rho_2(v_2 - D + D)(v_2 - D) - \rho_1(v_1 - D + D)(v_1 - D), \\
&= (\rho_2(v_2 - D) + D\rho_2)(v_2 - D) - (\rho_1(v_1 - D) + D\rho_1)(v_1 - D), \\
&= \rho_2(v_2 - D)^2 + D\rho_2(v_2 - D) - \rho_1(v_1 - D)^2 - D\rho_1(v_1 - D), \\
&= \rho_2(v_2 - D)^2 - \rho_1(v_1 - D)^2 + D\rho_2(v_2 - D) - D\rho_1(v_1 - D), \\
&= [\rho(v - D)^2] + D[\rho(v - D)], \\
&= [\rho(v - D)^2], \qquad ([\rho(v - D)] = 0).
\end{aligned}
$$

The second equation of (B.1) becomes

$$
\begin{aligned}
[p] + [\rho(v - D)^2] &= 0, \\
[p + \rho(v - D)^2] &= 0.
\end{aligned} \tag{B.4}
$$

For the third equation of (B.1) we use

$$
e = \frac{1}{\gamma - 1} \, \frac{p}{\rho},
$$

and

$$\begin{aligned}
[v^2] &= {v_2}^2 - {v_1}^2, \\
&= {v_2}^2 - 2Dv_2 + D^2 + 2Dv_2 - {v_1}^2 + 2Dv_1 - D^2 - 2Dv_1, \\
&= (v_2 - D)^2 + 2Dv_2 - (v_1 - D)^2 - 2Dv_1, \\
&= (v_2 - D)^2 - (v_1 - D)^2 + 2D(v_2 - v_1), \\
&= [(v - D)^2] + 2D[v],
\end{aligned} \tag{B.5}$$

Then the third equation of (B.1) becomes

$$\begin{aligned}
D[\rho(e + \frac{v^2}{2})] &= [v(p + \rho(e + \frac{v^2}{2}))], \\
D[\rho(\frac{1}{\gamma - 1}\ \frac{p}{\rho} + \frac{v^2}{2})] &= [v(p + \rho(\frac{1}{\gamma - 1}\ \frac{p}{\rho} + \frac{v^2}{2}))], \\
D[\frac{p}{\gamma - 1} + \rho\frac{v^2}{2}] &= [v(\frac{\gamma p}{\gamma - 1} + \rho\frac{v^2}{2})], \\
D[\frac{p}{\gamma - 1} + \rho\frac{v^2}{2}] &= [(v - D)(\frac{\gamma p}{\gamma - 1} + \rho\frac{v^2}{2}) + D(\frac{\gamma p}{\gamma - 1} + \rho\frac{v^2}{2}),] \\
D[\frac{p}{\gamma - 1} - \frac{\gamma p}{\gamma - 1}] &= [(v - D)(\frac{\gamma p}{\gamma - 1} + \rho\frac{v^2}{2})], \\
D[-p] &= [(v - D)(\frac{\gamma p}{\gamma - 1} + \rho\frac{v^2}{2})], \\
D[\rho(v - D)^2] &= [\rho(v - D)(\frac{1}{\gamma - 1}\ \frac{p}{\rho} + \frac{v^2}{2})], \\
0 &= [\rho(v - D)(\frac{1}{\gamma - 1}\ \frac{p}{\rho} + \frac{v^2}{2})] - D[\rho(v - D)^2], \\
0 &= [\rho(v - D)(\frac{1}{\gamma - 1}\ \frac{p}{\rho} + \frac{v^2}{2})] - D[\rho(v^2 - 2vD)], \\
0 &= [\rho(v - D)(\frac{1}{\gamma - 1}\ \frac{p}{\rho} + \frac{v^2}{2})] - D(\rho v(v - D)), \\
0 &= [\rho(v - D)(\frac{1}{\gamma - 1}\ \frac{p}{\rho} + \frac{v^2}{2} - Dv], \\
0 &= [\frac{1}{\gamma - 1}\ \frac{p}{\rho} + \frac{(v - D)^2}{2}].
\end{aligned} \tag{B.6}$$

This finalizes the proof. □

B.2 Proof of Asserting in Chapter III

Assume that all the gas dynamics variables are functions of one dependent variable ρ,

$$V(\rho), \quad p = P(\rho).$$

Substituting them into the gas dynamics equations, we obtain an overdetermined system of partial differential equations which in the matrix form is

$$\mathbf{A}\begin{pmatrix} \rho_t \\ \rho_x \end{pmatrix} = 0,$$

where

$$\mathbf{A} = \begin{pmatrix} 1 & v + \rho v_\rho \\ \rho v_\rho & \rho v v_\rho + P' \\ P' & vP' + \gamma P v_\rho \end{pmatrix}.$$

To show that $v + \frac{2\alpha}{\gamma-1}$ is constant in Riemann wave.

Proof. Note that

$$\gamma P - \rho(\rho v_\rho)^2 = 0,$$

$$\rho P' = \gamma P.$$

Consider the second equation

$$\rho P' = \gamma P.$$

Then we obtain

$$P = c_0\, \rho^\gamma,$$

where c_0 is an arbitrary constant.

Since

$$P = A(S)\, \rho^\gamma,$$

then

$$A(S) \;=\; \frac{P}{\rho^{\gamma}} \;=\; c_0,$$

Therefore

$$\frac{P}{\rho^{\gamma}} \;=\; c_2,$$

where c_2 is a constant.

Next, consider the first equation

$$v_\rho \;=\; \pm\; \frac{c}{\rho}$$

As

$$c \;=\; \sqrt{\frac{\gamma p}{\rho}} \qquad =\; \sqrt{\frac{\gamma A(S)\rho^{\gamma}}{\rho}} \qquad =\; \sqrt{\gamma A(S)}\rho^{\frac{\gamma-1}{2}}$$

and

$$\frac{P}{\rho^{\gamma}} = c_2.$$

Thus

$$\begin{aligned} v_\rho \;&=\; \pm\; \frac{c}{\rho} \\ &=\; \pm\; \frac{\sqrt{\gamma A(S)}\;\rho^{(\gamma-1)/2}}{\rho} \\ &=\; \pm\; \sqrt{\gamma A(S)}\rho^{\frac{\gamma-1}{2}-1}. \end{aligned}$$

Then integrate by $d\rho$,

$$\begin{aligned} v &= \pm\sqrt{\gamma A(S)}\left(\frac{2}{\gamma-1}\;\rho^{\frac{\gamma-1}{2}}\right) + c_1 \\ &= \pm\frac{2}{\gamma-1}\;\sqrt{\gamma A(S)\rho^{\gamma-1}} + c_1 \\ &= \pm\frac{2}{\gamma-1}\;\sqrt{\frac{\gamma P}{\rho}} + c_1 \\ &= -\frac{2\;\alpha}{\gamma-1} + c_1 \end{aligned}$$

where

$$\alpha = \mp\sqrt{\frac{\gamma P}{\rho}},$$

or equivalently,

$$\alpha^2 = \frac{\gamma P}{\rho}.$$

Therefore

$$v + \frac{2\ \alpha}{\gamma - 1} = c_1$$

and

$$\frac{P}{\rho^\gamma} = c_2.$$

□

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