CHAPTER III
MODELLING HEAVY-ION COLLISIONS

QCD, or Quantum Chromodynamics has limitations when it comes to describing the exact solution of quark and gluon dynamics and particle production within the heavy-ion collisions. For example, due to the asymptotic freedom (Gross and Wilczek, 1973) the perturbative QCD is not allowed and only feasible at higher energies (Politzer, 1973). In addition, in the heavy-ion collisions, the system needs to account for an enormous many-body problem and with different degrees of freedom at that, from parton to hadron. This makes the first-principle calculations practically impossible for a complete dynamical picture of the whole evolution.

However, one can still calculate these dynamics with an innovative approach like lattice QCD simulations (Borsanyi et al., 2014) for the non-perturbative QCD. These methods, however, have their own limitations in terms of computational complexity and reliable results are currently limited to relatively low net-baryon densities, $\mu_B/T \leq \pi$ (Allton et al., 2005; Aoki et al., 2006; de Forcrand and Philipsen, 2010; Bazavov et al., 2012; Vovchenko et al., 2018b; Philipsen, 2021). Hence, theoretical models are required for further development in terms of empirical models and/or phenomenological approaches.

Especially with the hunts for the critical point and the Equation of State (EoS) toward lower energy heavy-ion collisions, these studies are driven forces for even more rigorous improvement of these models. One of the alternative approaches, to describe the many-body behavior of the Quark-Gluon Plasma (QGP) and/or the hadronic interactions in heavy-ion collisions, is to look into the kinetic theory. This leads to the development of both transport models and hydrodynamics models.

Transport models rely on a microscopic description, treating each hadron individually via interactions and scattering processes. Hydrodynamics models, on the other hand, adopt a macroscopic viewpoint, treating the whole system as a fluid characterized by bulk properties like pressure, temperature, and density profile as well as transport coefficients. The EoS plays an important role in dictating the dynamics based on the relationship between these thermodynamic quantities. Both approaches have their own advantages and limitations.

In this chapter, we briefly review transport models with a special focus on the Ultra-relativistic transport model (UrQMD) and the hybrid model which combine the hydrodynamical description at the earlier state with the transport model by following the explanation in Ref. (Bratskovkaya, 2019a; Bratskovkaya, 2019b; Xu, 2019; Wolter et al., 2022; Sorensen et al., 2024).

3.1 Transport models

Every transport model for heavy-ion collisions begins with the kinetic theory that describes the time evolution of the N-body phase-space distribution function ρ by N-body Hamiltonian. Derived from the conservation law of the phase-space density $\frac{\mathrm{d}\rho}{\mathrm{dt}} = 0$, we can get the continuity equation or namely the Liouville equation reads as,

$$\frac{\partial \rho}{\partial \mathrm{t}} = \left\{\mathrm{H}, \rho\right\} . \tag{3.1}$$

This equation describes the free streaming of a single or N-body density according to the Hamiltonian H with $\left\{\cdot\right\}$ is the Poisson bracket. However, the dynamics in the heavy-ion collisions are far more complex, not only we have a huge multiplicity of the different particle species but they can also interact quantum mechanically, e.g. elastic collisions, inelastic collisions, and decays. Defining this N-body phase-space density is challenging and their interactions will appear as a gain and loss term for the phase-space density. Thus we need to derive the equation of motion that incorporate these quantum effects and express them in terms of quantum operators.

3.2 Boltzmann(Vlasov)-Uehling-Uhlenbeck (B(V)UU) approach

We begin our derivation with the N-body non-relativistic Schrödinger's equation describing the total wavefunction $\Psi\left(1, \ldots, \mathrm{N}; \mathrm{t}\right)$ with the Hamiltonian operator $\mathrm{H}_{\mathrm{N}}\left(1, \ldots, \mathrm{N}; \mathrm{t}\right)$ of N-particles,

$$\mathrm{i}\hbar \frac{\partial}{\partial \mathrm{t}} \Psi_{\mathrm{N}}\left(1, \ldots, \mathrm{N}; \mathrm{t}\right) = \mathrm{H}_{\mathrm{N}}\left(1, \ldots, \mathrm{N}; \mathrm{t}\right) \Psi_{\mathrm{N}}\left(1, \ldots, \mathrm{N}; \mathrm{t}\right) \tag{3.2}$$

By introducing the N-particle density matrix from the product of N-body wavefunctions, $\rho_{\mathrm{N}}\left(1, \ldots, \mathrm{N}; 1', \ldots, \mathrm{N}'; \mathrm{t}, \mathrm{t}'\right) = \Psi_{\mathrm{N}}^{*}\left(1, \ldots, \mathrm{N}; \mathrm{t}\right) \Psi_{\mathrm{N}}\left(1, \ldots, \mathrm{N}; \mathrm{t}\right)$, we can write Eq. 3.2 and its conjugate equation in the density matrix formalism, i.e., $\mathrm{i}\hbar \frac{\partial}{\partial \mathrm{t}} \rho_{\mathrm{N}} = \mathrm{H}_{\mathrm{N}} \rho_{\mathrm{N}}$

and $-i\hbar\frac{\partial}{\partial t'}\rho_N = H'_N\rho_N$. If we subtract the former with the latter equations and assume that $t = t'$, we finally arrive at the von-Neumann equation,

$$i\hbar\frac{\partial}{\partial t}\rho_N - \left[H_N, \rho_N\right] = 0 \ . \tag{3.3}$$

However, the determination of the N-body density as well as calculating the equation of motion for each particle is impractical. We need to reduce the N-body density matrix down to the desirable level. By taking the trace over particle $n+1^{th}$ to N^{th} of ρ_N, the n-body density is realized $\rho_n = \frac{1}{(N-n)!}Tr_{n+1,\ldots,N}\left(\rho_N\right)$. This reduced density matrix method leads to the BBGKY (Bogoliubov-Born-Green-Kirkwood-Yvon) hierarchy (Boercker and Dufty, 1979). This hierarchy provide a set of coupled equations to recursively determine the dynamics of the reduced n-body density matrices using the (n+1)-body density. The BBGKY hierarchy reads,

$$i\hbar\frac{\partial}{\partial t}\rho_n - \left[\sum_i^n H_i^0, \rho_n\right] = \left[\sum_{i<j}^{n-1} V_{ij}, \rho_n\right] + Tr_{n+1}\left(\left[\sum_{i=1}^{n} V_{i(n+1)}, \rho_{n+1}\right]\right) \ . \tag{3.4}$$

H_i^0 is the Hamiltonian of free n-particles (propagate within an external field). The second term represents the mutual interactions between particle i^{th} and others in the system. The third term is accounting for the correlations controlled by another (n+1)-particle.

The first two levels for the hierarchy equations read as,

$$i\hbar\frac{\partial}{\partial t}\rho_1 - \left[H_1^0, \rho_1\right] = Tr_2\left(\left[V_{12}, \ \rho_2\right]\right) \tag{3.5}$$

$$i\hbar\frac{\partial}{\partial t}\rho_2 - \left[H_1^0 + H_2^0, \rho_1\right] = \left[V_{12}, \rho_2\right] + Tr_3\left(\left[V_{12} + V_{13}, \ \rho_3\right]\right) \tag{3.6}$$

The calculation of the 2-body density matrix ρ_2 in the last term of Eq. 3.5 will be evaluated by the next hierarchy $\left(n = 2\right)$ in Eq. 3.6 which in turn also needs 3-body density ρ_3 from the next hierarchy, and so on. However, one can do some approximation to truncate the 2- or 3-body density down to the product of single-particle densities and close the set of equations. The different truncation schemes will lead to the different single-density equations of motions as shown in Ref. (Aichelin, 1991)

Time-Dependent Hartree-Fock (TDHF) approach: For instance, the Time-

Dependent Hartree-Fock (TDHF) approach for simple two-body density truncation which is limited to the effective in-medium mean field interactions. The 2-body density can be reduced into a product of antisymmetric product of single-particle density (fermion),

$$\rho_2\left(1, 2; 1', 2'; t\right) \approx A_{12}\rho_1\left(1, 1'; t\right)\rho_1\left(2, 2'; t\right),$$

where A_{ij} is the permutation operator. By substitute this back to Eq. 3.5, we obtain,

$$\begin{aligned} i\hbar\frac{\partial}{\partial t}\rho_1\left(1, 1'; t\right) - \left[T_1^0 - T_{1'}^0\right]\rho_1\left(1, 1'; t\right) + \left[V_1^0 - V_{1'}^0\right]\rho_1\left(1, 1'; t\right) \\ = Tr_2\left(\left[V_{12}^F A_{12} - V_{1'2'}^F A_{1'2'}\right]\right)\rho_1\left(1, 1'; t\right)\rho_1\left(2, 2'; t\right) \end{aligned} \tag{3.7}$$

From the formula, a single-particle i^{th} dynamics within TDHF approximation can be understood as a single-particle propagation with the kinetic T_i^0 under a direct influence within the medium (external fields and/or other nucleons). This interaction is determined from a self-generated local mean-field potential called Hartree term V_i^0 or V_i^H. Finally, a time-dependent exchange or non-local mean-field potential, called Fock potential V_i^F, accounts for the Pauli principle.

In order to draw any further meaningful interpretation from the density matrix ρ, we need to derive the proper phase-space distribution function $f\left(\vec{r}, \vec{p}\right)$. Hence, we will perform the Wigner transformation which is a Fourier transform on a single-particle wavefunction or single-particle density matrix (Hillery et al., 1984) with new variable $\vec{s} \equiv \vec{x}_1 - \vec{x'}_1$ and $\vec{r} \equiv \left(\vec{x}_1 + \vec{x'}_1\right)/2$ which reads,

$$f\left(\vec{r}, \vec{p}, t\right) = \int d^3s \exp\left(\frac{-i\vec{p}\cdot\vec{s}}{\hbar}\right)\rho\left(\vec{r} + \vec{s}/2, \vec{r} - \vec{s}/2\right) \tag{3.8}$$

For simplicity, we will determine the phase-space distribution from TDHF approach within the diluted gas limit, i.e., neglecting the Fock time-dependent exchange term.

Under the Wigner transformation, Eq. 3.7 becomes,

$$
\begin{aligned}
&i\hbar\frac{\partial}{\partial t}\int d^3s\,\exp\left(\frac{-ip\cdot s}{\hbar}\right)\rho\left(r+s/2,r-s/2\right)\\
&=\frac{\hbar^2}{2m}\int d^3s\,\exp\left(\frac{-ip\cdot s}{\hbar}\right)\left[\nabla^2_{r+s/2}-\nabla^2_{r+s/2}\right]\rho\left(r+s/2,r-s/2\right)\\
&+\int d^3s\,\exp\left(\frac{-ip\cdot s}{\hbar}\right)\left[V^0\left(r+s/2\right)-V^0\left(r-s/2\right)\right]\rho\left(r+s/2,r-s/2\right)
\end{aligned}
\tag{3.9}
$$

Then we can simplify the second term with $\left[\nabla^2_{r+s/2}-\nabla^2_{r+s/2}\right]=2\nabla_r\cdot\nabla_s$, and the third term by the Taylor expansion around $s\to 0$ of self-consistent mean-field potentials $\lim_{s\to 0}\left[V^0\left(r+s/2\right)-V^0\left(r-s/2\right)\right]\approx s\nabla_r V^0=s\nabla_r U$. Finally, The Vlasov equation is obtained,

$$
\frac{\partial f}{\partial t}+\frac{p}{m}\nabla_r f+\nabla_r U\cdot\nabla_p f=0\,. \tag{3.10}
$$

The equation describes a free propagation of a single-particle phase-space distribution $f\left(r,p\right)$ in the self-generated Hartree mean-field potential U or U_H. The right hand side of the Vlasov equation can be non-zero, if we introduce the quantum effect corrections, e.g., Fock exchange potential or the collision term $\left(\frac{\partial f}{\partial t}\right)_{coll}$.

Vlasov-Uehling-Uhlenbeck (VUU) approach: If one wants to include the potential from the realistic interaction between particles instead of the effective mean-field potential. We need to evaluate the dynamics of a single-particle from, at least, a truncation of the 3-body density matrices. The coupling term with a second particle is responsible for the collision with the first particle. While the third particle will generate an external field to correlate with the first two particles. However, this requires lengthy mathematics calculations where one needs to discuss not only the reduction of the density but also the calculation that arises from the trace terms, e.g., Pauli-blocking and G-matrix, which is not really inline with our focus in this thesis. Hence, we will only discuss how to derive the VUU equation with the collision term from the 2-body level density or second hierarchy of the BBGKY equation $n=2$.

The 2-body density can be reduced into single-particle densities while non-zero two-body correlations are still kept into account. The 2-body density matrix be-

comes,

$$\rho_2(1,2;1',2';t) \approx A_{12}\rho_1(1,1';t)\rho_1(2,2';t) + c_2(1,2;1',2';t)\ .$$

Substitute this reduced density into the BBGKY second hierarchy Eq. 3.6, we get a similar result to the TDHF approach with an additional trace term of c_{12},

$$\overbrace{i\hbar\frac{\partial}{\partial t}\rho_1(1,1';t) - \left[H_1^0 - H_{1'}\right]\rho_1(1,1';t)}^{\text{TDHF-Vlasov}} = \underbrace{\mathrm{Tr}_2\left(\left[V_{12} - V_{1'2'}\right]\right)c_2(1,2;1',2';t)}_{\text{Collision term + Pauli exchange}} \tag{3.11}$$

Finally, we perform the Wigner transformation to express the equation in terms of physical phase-space distribution. This equation of motion is called "VUU" or "BUU" equation;

$$\frac{\partial f}{\partial t} + \frac{\vec{p}}{m}\vec{\nabla}_r f - \vec{\nabla}_r U \cdot \vec{\nabla}_p f = I_{coll}\left[f_1, \ldots, f_N\right]. \tag{3.12}$$

The left hand side is equivalent to the Vlasov equation describing the free propagation within the mean-field where U is a self-consistent potential. On the right hand side, I_{coll} represents the collision integral arising from the trace term with the correlation matrix density. This term is connected to the transition rates from various contributed collision processes which reads,

$$I_{coll} = \frac{1}{(2\pi)^3}\int d^3\vec{p}_2 d^3\vec{p}_3 d\Omega |v_{12}| \left(\frac{d\sigma}{d\Omega}\right)_{12\,\to\,34} \delta^3(\vec{p}_1 + \vec{p}_2 - \vec{p}_3 - \vec{p}_4)$$
$$\times\Big[\underbrace{f_3 f_4(1-f_1)(1-f_2)}_{\text{Gain term: } 3+4\,\to\,1+2} - \underbrace{f_1 f_2(1-f_3)(1-f_4)}_{\text{Loss term: } 1+2\,\to\,3+4}\Big]. \tag{3.13}$$

Several BUU-based transport models have been developed to simulate heavy-ion collisions, e.g., GiBUU (The Giessen BUU model) (Buss et al., 2012), PHSD (The Parton-Hadron-String Dynamics model) which includes both hadronic and partonic degree of freedom (Linnyk et al., 2016), and SMASH (The Simulating Many Accelerated Strongly-interacting Hadrons model) (Weil et al., 2016). In BUU transport models, the phase-space distribution function is often represented using test particles (Wong, 1982). This method discretizes the continuous distribution function into a large number of test

particles, each representing a small fraction of the system:

$$f_i(r, p, t) \sim \frac{1}{N_{tp}} \sum_i g(r - r_i(t) g(p - p_i(t)) \quad (3.14)$$

Here, N_{tp} is the number of the test particles and the g is the sharp function, e.g., the delta function. These test particles are generated for every time step and averaged over all events. Also, the mean-field potential is also updated according to the one-body test particles. However, this method reduces the ability to generate realistic 2-body correlations in each time step as well as the event-by-event fluctuations. These issues are, however, not a problem with the QMD approach which we will discuss later.

3.3 Ultra-relativistic Quantum Molecular Dynamics (UrQMD) model

In contrast to BUU models, Quantum Molecular Dynamics (QMD) approaches simulate the system using classical equations of motion for quasi-particles (Aichelin, 1991). Instead of using the test particles, each particle is represented as a Gaussian wave packet (Ono et al., 1992a), and their dynamics are governed by mutual 2-body interactions through effective potentials and realistic binary particle collisions. Thus the QMD approach (Hartnack et al., 1989; Ono et al., 1992a; Bass et al., 1998; Bleicher et al., 1999; Nara et al., 2000; Aichelin et al., 2020) could directly provide many-body correlations and fluctuations. It has recently extended its capabilities to describe larger clusters and even hyperclusters. In this thesis, we utilize the latest version of the UrQMD transport model (UrQMD v3.5). The UrQMD will be briefly introduced in this section.

3.3.1 Initialization

QMD approach initializes the particles with the explicit N-body nuclear wavefuction which can be approximated with a simple product of single-particle wavefunction, $\Psi(\vec{r}; t) = \prod_i^N \psi_i(\vec{r}, \vec{r_i}; t)$. Note that, in UrQMD, we can neglect the Slater determination for the antisymmetrization from the Pauli in exchange for the computational time. However, the nature of the two-body correlation will arise from the effective Pauli potential which will be discussed below. However, a few QMD models attempt to initialize with real antisymmetric wavefunction, e.g., Antisymmetrized Molecular Dynamics (AMD) model (Ono et al., 1992b) and the Fermionic Molecular Dynamics

(FMD) model (Feldmeier, 1990). In UrQMD, the single-particle wavefunction is assumed as a Gaussian wave packet,

$$\psi_i(\vec{r}, \vec{r_i}; t) = \left(\frac{1}{2\pi L}\right)^{4/3} \exp\left[-\frac{(\vec{r} - \vec{r_i}(t))^2}{4L} - i\vec{p_i}(t)\cdot\vec{r}\right] . \quad (3.15)$$

L is the width of the Gaussian wave packet in coordinate space. Then, we apply the Wigner transform in a similar fashion from Eq. 3.8. The phase-space density of particle i^{th} reads,

$$f_i(\vec{r}, \vec{p}; t) = \left(\frac{1}{\pi\hbar}\right)^3 \exp\left[-\frac{(\vec{r} - \vec{r_i}(t))^2}{2L} - 2L(\vec{p} - \vec{p_i}(t))\right] . \quad (3.16)$$

The equation of motion in the QMD approach is derived from the Euler-Lagrange. The generalized Lagrangian function is given by,

$$\mathcal{L} = \int d^3r_1 \ldots d^3r_N \Psi^* \left(-i\frac{\partial}{\partial t} - H\right) \Psi, \quad (3.17)$$

$$\mathcal{L} = \left(\frac{d\vec{r_i}}{dt} \cdot \vec{p_i} - T_i - \sum_j \langle V_{ij}^{tot} \rangle - \frac{3}{8mL}\right) . \quad (3.18)$$

The N-body Hamiltonian is defined as $H = \sum_i T_i + \sum_{j \neq i} V_{ij}^{tot}$ where the kinetic energy and the total potential energy of the particle i^{th} are T_i and V_{ij}^{tot} with $\langle V_{ij}^{tot} \rangle = \int d^3r_i d^3p_i \int d^3r_j' d^3p_j' f_i(\vec{r}, \vec{p}) V_{ij} f_j(\vec{r'}, \vec{p'})$. The total 2-body interactions involved

in the propagation are the following:

$$V^{Skyrme} = \alpha\left(\frac{\rho}{\rho_0}\right) + \beta\left(\frac{\rho}{\rho_0}\right)^{\gamma}, \tag{3.19}$$

$$V^{Yukawa} = C_{yuk}\frac{\exp\left(-\mu|\vec{r_i} - \vec{r_j}|\right)}{|\vec{r_i} - \vec{r_j}|}, \tag{3.20}$$

$$V^{Coul} = \frac{Z_i Z_j e^2}{|\vec{r_i} - \vec{r_j}|}, \tag{3.21}$$

$$V^{MD} = t_4 \ln\left[t_5\left(\vec{p_i} - \vec{p_j}\right)^2 + 1\right]\delta\left(\vec{r_i} - \vec{r_j}\right), \tag{3.22}$$

$$V^{Pauli} = \frac{C_{pauli}}{\left(q_0 p_0\right)^3}\exp\left[-\frac{\left(\vec{r_i} - \vec{r_j}\right)^2}{2q_0^2} - \frac{\left(\vec{p_i} - r\vec{p_j}\right)^2}{2p_0^2}\right]. \tag{3.23}$$

Already from here, one can see the difference between the QMD and BUU for the treatment of the 2-body (and 3-body) interactions. In QMD, they appear as effective potential replacing the real part of the transition matrix from the collision term. The local short-range V^{Skyrme} and and long-range V^{Yukawa} potentials are responsible for the nucleon-nucleon interaction in the nucleus. The parameters α, β, γ are fixed to mimic the proper binding energy and the compressibility ruling the stiffness of the density-dependent equation of state, e.g., soft EoS and hard EoS. The parameter C_{yuk} is necessary to ensure the nuclear surface properties and its fluctuations. The Coulomb potential V^{Coul} is straightforward and does not need further explanation. The Momentum-Dependent potential V^{MD} is a small correction for more repulsive nuclear interactions leading to a small adjustment for the Skyrme's parameters and the stiffness of the equation of state. Finally, The Pauli potential is parametrized to prevent fermions from occupying the same phase-space cell. The last two potentials, although optional, are important in order to investigate the nuclear properties at low energies where the two-body or three-body correlations become strong, e.g., collective flow (Steinheimer et al., 2018; Steinheimer et al., 2022) and fluctuations (Ye et al., 2020).

3.3.2 Propagation and Collision

The propagation of all particles is also based on the non-relativistic Boltzmann or BUU equations similar to Eq. 3.12 where the time evolution of the momentum and

coordinate of particle i^{th} is evaluated by the Euler—Lagrange equations,

$$\frac{d\vec{r_i}}{dt} = \frac{\partial\langle H\rangle}{\partial\vec{p_i}}, \tag{3.24}$$

$$\frac{d\vec{p_i}}{dt} = \frac{\partial\langle H\rangle}{\partial\vec{r_i}}\ . \tag{3.25}$$

However, unlike the BUU approach, UrQMD does not operate with the phase-space density or test particles but real microscopic n-body (classical-)particles of 70 baryon- and 39 meson-species (Table 3.1 and 3.2) and their antiparticles. Hence, the scattering process in the collision (source) term between hadrons will be triggered based on a geometrical interpretation of elastic and inelastic cross-sections. The collision will occur once the relative distance between two particles is below $d \leq \sqrt{\sigma_{tot}/\pi}$.

Furthermore, while two-body or three-body interactions in the BUU approach are calculated together in the collision term I_{coll}. It requires tedious calculation for both the real and imaginary parts of the transition matrix (or the Bruckner g-matrix). However, in the UrQMD approach, these interactions are expressed in terms of effective potentials replacing the real part of the transition matrix and leaving only the imaginary part in the collision terms. The imaginary part of scattering amplitude is modeled via binary elastic and inelastic collisions and fitted with the experimental data.

Baryon-Baryon scattering: The cross-section of the BB-scattering is given by,

$$\sigma_{tot}^{BB}\left(\sqrt{s}\right) \propto \left(2S_1+1\right)\left(2S_2+1\right)\frac{\langle p_{3,4}\rangle}{\langle p_{1,2}\rangle}\frac{1}{s}\left|\mathcal{M}\left(m_3, m_4\right)\right|^2, \tag{3.26}$$

where S_i is the spin of particle i^{th}. The matrix element $\left|\mathcal{M}\left(m_3, m_4\right)\right|$ will be taken in a general form with an adjustable free parameter to fit with the experiments. Or if the out-going particles are resonances, the matrix element will be written according to the mass distributions of Breit-Wigner form and fit with the decay width and branching ratio from PDG.

Meson scattering: The scattering processes for the meson are mostly reso-

Table 3.1 Table of Baryons.

N	Δ	Λ	Σ	Ξ	Ω
N_{938}	Δ_{1232}	Λ_{1116}	Σ_{1192}	Ξ_{1317}	Ω_{1672}
N_{1440}	Δ_{1600}	Λ_{1405}	Σ_{1385}	Ξ_{1530}	
N_{1520}	Δ_{1620}	Λ_{1520}	Σ_{1660}	Ξ_{1690}	
N_{1535}	Δ_{1700}	Λ_{1600}	Σ_{1670}	Ξ_{1820}	
N_{1650}	Δ_{1900}	Λ_{1670}	Σ_{1750}	Ξ_{1950}	
N_{1675}	Δ_{1905}	Λ_{1800}	Σ_{1775}	Ξ_{2025}	
N_{1680}	Δ_{1910}	Λ_{1810}	Σ_{1915}		
N_{1700}	Δ_{1920}	Λ_{1820}	Σ_{1940}		
N_{1710}	Δ_{1930}	Λ_{1830}	Σ_{2030}		
N_{1720}	Δ_{1950}	Λ_{1890}			
N_{1900}		Λ_{2100}			
N_{1990}		Λ_{2110}			
N_{2080}					
N_{2190}					
N_{2200}					
N_{2250}					

Table 3.2 Table of Mesons.

0^{0-}	1^{--}	0^{++}	1^{++}
π	ρ	a_0	a_1
K	K^*	K_0^*	K_1^*
η	ω	f_0	f_1
η'	ϕ	f_0^*	f_1^*
1^{+-}	2^{++}	$\left(1^{--}\right)^*$	$\left(1^{--}\right)^*$
b_1	a_2	ρ_{1450}	ρ_{1700}
K_1	K_2^*	K_{1410}^*	K_{1680}^*
h_1	f_2	ω_{1420}	ω_{1662}
h_1'	f_2'	ϕ_{1680}	ϕ_{1900}

nance scattering. For example, the total cross-section for MB-scattering reads,

$$\sigma_{tot}^{MB}\left(\sqrt{s}\right) = \sum_{R=\Delta,N^*} \langle j_B, m_B, j_M, m_M \| J_R, M_R \rangle \frac{2S_R+1}{\left(2S_B+1\right)\left(2S_M+1\right)} \times \frac{\pi}{p_{cm}^2} \frac{\Gamma_{R\to MB}\Gamma_{tot}}{\left(M_R-\sqrt{s}\right)^2+\Gamma^2/4} \tag{3.27}$$

where the decay width and branching ratio are obtained from the PDG data. However, for the unknown cross-sections, e.g., $\bar{B}B$, B^*B, YN, M^*B^*, M^*M^*, these processes will be obtained via detailed balance (Goulianos, 1983) or the Additive Quark Model (AQM) re-scaling (Danielewicz and Bertsch, 1991). For example,

$$\sigma_{tot}^{MM}\left(\sqrt{s} > 1.7\ \text{GeV}\right) = \sigma_{tot}^{\pi p}\left(\sqrt{s}\right) \frac{\sigma_{AQM}^{MM}}{\sigma_{AQM}^{\pi p}} \,. \tag{3.28}$$

One also should be noted that the particle productions are not only subjected to the scattering processes or cross-section calculations. The string excitation, fragmentation, and coalescence routines are also embedded to the current version of the UrQMD v3.5. The detailed implementations for the string excitation and the fragmentation can be found in Ref. (Andersson et al., 1983). While the coalescence routine will be discussed here.

3.3.3 (Hyper)nuclei Formation Routine

Another remarkable update of the UrQMD v3.5 is a coalescence routine for the (hyper)cluster formations. The coalescence model states that if two or more constituent particles are close enough in phase-space, they will coalesce and form into a cluster (see Ch. V for more details). Within UrQMD v3.5, the coalescence model is applied numerically using a method known as box coalescence.

For each pair of nucleons and/or hyperons, the relative distance in their center of mass frame is calculated. If their relative distance $\Delta x = |\mathbf{x}_{n,1} - \mathbf{x}_{n,2}|$ is less than $\Delta x_{max,nn}$ and their relative momentum $\Delta p = |\mathbf{p}_{n,1} - \mathbf{p}_{n,2}|$ is less than $\Delta p_{max,nn}$, and if the spin-isospin coupling probabilities are satisfied, a two-body state will form with combined momentum $\mathbf{p}_{nn} = \mathbf{p}_{n,1} + \mathbf{p}_{n,2}$ and position $\mathbf{x}_{nn} = \left(\mathbf{x}_{n,1} + \mathbf{x}_{n,2}\right)/2$.

The procedure is then extended to form three-body cluster states. The relative distance within their local rest frame $\Delta x = |\mathbf{x}_{nn,12} - \mathbf{x}_{n,3}|$ must be

less than $\Delta x_{max,nnn}$, and the relative momentum $\Delta p = |\mathbf{p}_{nn,12} - \mathbf{p}_{n,3}|$ must be less than $\Delta p_{max,nnn}$. If these conditions are metand the appropriate spin-isospin probabilities are satisfied, a three-body cluster will form with combined momentum $\mathbf{p}_{nnn} = \mathbf{p}_{nn,12} + \mathbf{p}_{n,3}$ and position $\mathbf{x}_{nnn} = \left(\mathbf{x}_{n,1} + \mathbf{x}_{n,2} + \mathbf{x}_{n,3}\right)/3$.

The criteria for forming these clusters are listed in Table **??**, where Δx_{max} and Δp_{max} are determined based on iterative fits to cluster multiplicity data from various experiments (see Ref. (Hillmann et al., 2018; Sombun et al., 2019) for light nuclei and Ref. (Reichert et al., 2023c; Reichert et al., 2023d) for hypernuclei).

Table 3.3 The numerical coalescence parameters of UrQMD v3.5.

Particle	Δx [fm]	Δp [GeV/c]	spin-isospin
d	4.0	0.25	3/8
t, ^{3}He	3.5	0.32	1/12
^{4}He	3.5	0.41	1/12
NΞ	9.5	0.15	3/8
$^3_\Lambda$H	9.5	0.15	1/12

In conclusion, due to the realistic n-body treatments of the UrQMD, the coalescence model will also register two-body correlation effects throughout the evolution allowing us to better capture the fluctuations, the correlations, and cluster formation (which may relate to the critical behavior) than other transport approaches. All these aspects are inline with our following investigations, thus we will use UrQMD v3.5 as our basis for the event simulations.

3.4 Hydrodynamics Models

The hydrodynamics model offers a comprehensive framework for simulating the dynamics of heavy-ion collisions, providing insights into the evolution of the quark-gluon plasma (QGP). One of the key advantages of hydrodynamic models is the explicit description of the thermodynamic properties of the medium, allowing for a more accurate incorporation of the equation of state (EoS).

The conservation laws of energy-momentum and baryon density form the

foundation of hydrodynamic models, expressed by the following equations:

$$\partial_\mu T^{\mu\nu} = 0, \quad \text{(Energy-Momentum Conservation)}$$
$$\partial_\mu N^{\mu} = 0, \quad \text{(Baryon Four-Current Conservation)}$$

Here, $T^{\mu\nu}$ represents the energy-momentum tensor, and N^{μ} denotes the baryon four-current. These quantities can be expressed in terms of the fluid's four-velocity u^{μ} and the thermodynamic state in the local rest frame of the fluid, described by the energy density ϵ, the pressure p, and the baryon density n:

$$T^{\mu\nu} = (\epsilon + p)u^{\mu}u^{\nu} - pg^{\mu\nu}, \quad N^{\mu} = nu^{\mu}. \tag{3.29}$$

In addition to the hydrodynamic equations, a specific EoS, $p = p(\epsilon, n)$, is required to close these coupling equations.

The standard relativistic hydrodynamics model treats the medium from the collision as a single fluid (Belenkij and Landau, 1955; Amsden et al., 1975; Wong et al., 1975; Csernai et al., 1980; Mishustin et al., 1987; Rischke, 1999; Spieles and Bleicher, 2020). However, this simple approach has its limitations, particularly in heavy-ion collisions where the conditions are far from equilibrium (Noronha-Hostler et al., 2016). One of the major shortcomings of the single fluid model is the assumption of instantaneous thermalization and infinite stopping power, which leads to unrealistically rapid thermal equilibration and excessive energy densities.

To address these issues, three-fluid hydrodynamics models have been proposed (Mishustin et al., 1989; Katscher et al., 1993; Brachmann et al., 1997; Ivanov et al., 2006; Batyuk et al., 2016; Cimerman et al., 2023). These models aim to parametrize the projectile, target, and fireball as separate fluids within the system. This three-fluid approach allows for a more realistic initialization and handling of the hydrodynamic expansion stage of the collisions. Each fluid component can permeate and exchange energy-momentum appropriately, overcoming the limitations of the single-fluid model. The basic equations of relativistic hydrodynamics for the three-fluid model involve the conservation of energy-momentum and baryon current for each fluid i:

$$\partial_\mu T_i^{\mu\nu} = F_i^{\nu}, \quad \partial_\mu j_i^{\mu} = S_i \quad (i = 1, 2, 3)$$

where the source terms F_i^{ν} and S_i account for the energy-momentum and baryon charge

exchange between the fluids. The total conservation equations obey $\sum_i F_i^{\nu} = 0$ and $\sum_i S_i = 0$. These source terms include frictional forces arising from the relative motion between fluids, where viscosity could enter the equations (Busza et al., 2018), influencing the flow dynamics and leading to the development of shear and bulk viscosity effects.

Hydrodynamic models, as macroscopic descriptions, primarily provide macroscopic observables or bulk properties of heavy-ion collisions, such as thermodynamic properties that may not be directly measured, like particle spectra. To translate these thermodynamic quantities into experimental observables, the freeze-out condition must be defined, marking the transition where particles free stream to the detectors.

Defining the freeze-out condition within hydrodynamic models is not straightforward and requires certain assumptions to describe hadronization. The "particlization" approach, which is adopted from the Cooper-Frye approach, occurs when the local thermodynamic properties (energy density) of the medium at the freeze-out hypersurface fall below a certain threshold. At this point, the fluid elements convert into hadrons, which subsequently stream freely to the detectors.

However, the criteria for freeze-out are debatable and model-dependent. This has led to a newly novel approach suggesting a continuous freeze-out during the evolution instead of a snapshot of the hypersurface at the end of the calculation (Grassi et al., 1996; Hung and Shuryak, 1998; Akkelin et al., 2008; Knoll, 2009).

Another approach is the hybrid model, where the hydrodynamic model is integrated into the early stage of a microscopic transport model, often referred to as an afterburner. This method provides the realistic initial stage and the possible phase transition from QGP where the partons dominate the system to a hadronic phase as well as realistic dynamics and kinetic freeze-out during and after the hadronic phase, e.g., resonances, decays, and re-scattering.

3.5 Hybrid Models

Describing the hadronic freeze-out and extracting secondary particle spectra presents a challenge for hydrodynamic models. The application of the Cooper-Frye equation (Cooper and Frye, 1974) is commonly used for this purpose, where the transition from the fluid to free hadrons occurs instantaneously on a hypersurface of equal local energy density.

Pure microscopic models, such as transport models, excel at describing the hadronic phase of heavy-ion collisions, effectively capturing the non-equilibrium dynamics and individual particle interactions. However, they struggle to accurately describe the QGP or partonic phase and capture the effects of a phase transition in the thermodynamical perspective which is important for the EoS studies.

Conversely, macroscopic hydrodynamic models are adept at describing the QGP phase and the collective flow at the early stage of the collisions, assuming local thermal equilibrium and using EoS to describe the system's evolution. Yet, they fall short in describing the hadronic phase, particularly the freeze-out process and subsequent particle interactions, due to their reliance on equilibrium assumptions.

The hybrid model aims to combine the strengths of both microscopic transport models and macroscopic hydrodynamic models. By integrating these approaches, hybrid models provide a comprehensive description of the entire evolution of a heavy-ion collision, from the initial QGP phase to the final hadronic interactions. In this section, we will briefly introduce the implementation based on the UrQMD hybrid approach (Steinheimer et al., 2008; Steinheimer et al., 2012).

The UrQMD hybrid model incorporates the hydrodynamic evolution of the QGP phase allowing for a more accurate depiction of the transition from (local) equilibrium hydro phase to non-equilibrium hadronic phases. The implementation steps are the following:

- **Initial State:** The hydrodynamic evolution begins after the two Lorentz-contracted nuclei have passed through each other, triggering the thermalization. The energy and momentum of the participants are mapped into hydrodynamic quantities serving as initial conditions (Petersen et al., 2008). The initial geometry is typically based on the Glauber model.

- **Hydrodynamic Evolution and EoS:** A (3+1)-dimensional ideal hydrodynamic evolution is assumed for the early stage collision, where the system is thermalized. The chosen EoS guides the fluid's evolution, determining how the system transitions from QGP to hadronic matter.

- **Final State - Particlization:** The change from the hydrodynamic to transport phase occurs through particlization, where hydrodynamic parameters are converted to hadron distributions using the Cooper-Frye procedure (Cooper and Frye, 1974). Finally, these particles and their interactions, e.g., scattering and

resonance decays, are simulated within the UrQMD transport algorithm until freeze-out.

In addition to the UrQMD hybrid model, several other hybrid models have been developed, integrating various transport and hydrodynamic approaches:

- Integrated UrQMD 3.3 (Petersen et al., 2008)
- Hadronic Dissipative Effects (Hirano et al., 2006)
- 3-D Hydro + Cascade Model at RHIC (Nonaka and Bass, 2006)
- NeXSpheRIO (Hama et al., 2008)
- EPOS+Hydro+UrQMD at LHC (Werner et al., 2010)
- MUSIC@RHIC and LHC (Schenke et al., 2011)

As mentioned before, particlization in pure hydrodynamics is akin to freeze-out, where particle spectra are generated from the final stage of the hydrodynamic evolution. However, in the hybrid model, particlization refers to the mapping of hydrodynamic quantities to particle distributions for the transport model. It is important to note that particlization is neither the hadronization process nor freeze-out but rather a practice for transitioning between two frameworks while maintaining consistent initial and freeze-out conditions.

The particlization is implemented numerically with a Monte Carlo based on the Cooper-Frye equation:

$$E\frac{dN}{dp^3} = \int d\sigma_\mu p^\mu f(x, p) \approx \sum_\sigma \Delta\sigma_\mu p^\mu f(x, p) \,, \tag{3.30}$$

where σ_μ is a hypersurface element of hydrodynamics. The challenge lies in determining both the location and the normal direction of these surface elements. In Eq. (3.30), $f(x, p)$ represents the momentum distribution of hadrons inside the fluid. The assumptions about the fluid are directly reflected in the particle distribution

function. For near-equilibrium fluids, allowing for dissipation (viscous hydrodynamics), $f(x, p)$ reads:

$$f(x, p) = f_{F/B}(x, p) + \delta f\left(C_s^{(\text{shear})}, C_s^{(\text{bulk})}\right) . \tag{3.31}$$

This function is simply the density distribution of a fermion/boson gas, $f_{F/B}(x, p)$, plus terms representing shear and/or bulk viscosities, c_s (from dissipating non-equilibrium effects).